Midterm 1 Solutions

Math 124, Fall 2026

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Room: ◯ 1360 East Hall ◯ 3088 East Hall

Instructions

- This exam consists of 8 problems across 12 pages, worth a total of 100 points.
- You have 120 minutes to complete this exam, unless you have extended-time accommodations through SSD.
- Write your uniqname in the top right corner of each page where a space is provided.
- For free response problems, **you must show all of your work**, and write your final answer in the small box when one is provided.
- For multiple choice problems, completely fill in bubbles and square boxes.
  ◯ A bubble means that you should only select one choice.
  ☐ A square box means that you should select all that apply.
- You may refer to **one double-sided** 4×6**-inch index card**, handwritten directly on the card. Other than that, you may not refer to any other resources or technology during the exam (no phones, watches, or calculators).

By signing below, I affirm that this work is my own and that I have followed the exam's resource and collaboration rules.

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Problem 1 (10 pts)

Let

$$\vec{d} = \begin{bmatrix} 6 \\ -9 \\ 18 \end{bmatrix}$$

You may leave numbers as unsimplified fractions.

a) (4 pts) Find a unit vector pointing in the same direction as $\vec{d}$.

Solution: First factor out 3:

$$\vec{d} = \begin{bmatrix} 6 \\ -9 \\ 18 \end{bmatrix} = 3 \begin{bmatrix} 2 \\ -3 \\ 6 \end{bmatrix}$$

The length of the vector that remains is $\sqrt{2^2 + (-3)^2 + 6^2} = \sqrt{49} = 7$. So $\|\vec{d}\| = 3(7) = 21$. Dividing by the length gives the unit vector:

$$\frac{\vec{d}}{\|\vec{d}\|} = \frac{3}{21} \begin{bmatrix} 2 \\ -3 \\ 6 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 2 \\ -3 \\ 6 \end{bmatrix} = \boxed{\begin{bmatrix} 2/7 \\ -3/7 \\ 6/7 \end{bmatrix}}$$

b) (3 pts) Find a vector of length 6 pointing in the direction opposite to $\vec{d}$.

Solution: Multiply the unit vector from part (a) by -6. The factor 6 sets the length, and the negative sign reverses the direction:

$$-6\frac{\vec{d}}{\|\vec{d}\|} = -\frac{6}{21} \begin{bmatrix} 6 \\ -9 \\ 18 \end{bmatrix} = \boxed{\begin{bmatrix} -12/7 \\ 18/7 \\ -36/7 \end{bmatrix}}$$

c) (3 pts) A robot starts at $P = (5, -6, 8)$ and moves x units in the direction of $\vec{d}$, ending at $(9, -12, 20)$. What is the value of x?

Solution: The displacement from the starting point to the ending point is

$$\begin{bmatrix} 9-5 \\ -12-(-6) \\ 20-8 \end{bmatrix} = \begin{bmatrix} 4 \\ -6 \\ 12 \end{bmatrix} = \frac{2}{3}\vec{d}$$

The positive scalar confirms that the robot moves in the direction of $\vec{d}$. The distance traveled is the length of this displacement:

$$x = \left\| \frac{2}{3}\vec{d} \right\| = \frac{2}{3}(21) = \boxed{14}$$

Problem 2 (14 pts)

Suppose $\vec{u}, \vec{v} \in \mathbb{R}^3$ satisfy

$$\|\vec{u}\| = 3, \qquad \|\vec{v}\| = 4, \qquad \vec{u} \cdot \vec{v} = -10$$

a) (6 pts) Find the value of c for which $\vec{u} + c\vec{v}$ is orthogonal to $2\vec{u} - \vec{v}$.

Solution: Orthogonal vectors have dot product 0. Distribute the dot product and use $\vec{u} \cdot \vec{u} = \|\vec{u}\|^2$:

$$\begin{aligned} 0 &= (\vec{u} + c\vec{v}) \cdot (2\vec{u} - \vec{v}) \\ &= 2\|\vec{u}\|^2 - \vec{u} \cdot \vec{v} + 2c(\vec{u} \cdot \vec{v}) - c\|\vec{v}\|^2 \\ &= 2(9) - (-10) + 2c(-10) - 16c \\ &= 28 - 36c \end{aligned}$$

Therefore, $c = \dfrac{28}{36} = \boxed{\dfrac{7}{9}}$.

b) (5 pts) Find $\|3\vec{u} + \vec{v}\|$.

Solution: First find the squared length by taking a dot product:

$$\begin{aligned} \|3\vec{u} + \vec{v}\|^2 &= (3\vec{u} + \vec{v}) \cdot (3\vec{u} + \vec{v}) \\ &= 9\|\vec{u}\|^2 + 6(\vec{u} \cdot \vec{v}) + \|\vec{v}\|^2 \\ &= 9(9) + 6(-10) + 16 = 37 \end{aligned}$$

Taking the nonnegative square root gives $\boxed{\|3\vec{u} + \vec{v}\| = \sqrt{37}}$.

c) (3 pts) Find the angle between $\vec{u}$ and $\vec{v}$. Then select the angle type.

◯ Acute ◯ Right ● Obtuse

Solution: Using the formula for cosine similarity (which comes from the geometric definition of the dot product):

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|\|\vec{v}\|} = \frac{-10}{3(4)} = -\frac{5}{6}$$

Thus $\boxed{\theta = \cos^{-1}(-5/6)}$. The dot product is negative, so the angle is **obtuse** (greater than $90°$).

Problem 3 (15 pts)

Consider the line ℓ in $\mathbb{R}^2$ given by

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix} + t \begin{bmatrix} 6 \\ 5 \end{bmatrix}, \qquad t \in \mathbb{R}$$

a) (4 pts) Write an equation for ℓ in the form $ax + by = c$.

Solution: A normal $\begin{bmatrix} a \\ b \end{bmatrix}$ must be perpendicular to the direction $\begin{bmatrix} 6 \\ 5 \end{bmatrix}$, so $6a + 5b = 0$. An easy choice is $a = 5$ and $b = -6$; this comes from swapping 6 and 5 and negating one of them. Then, to find c in

$$5x - 6y = c$$

just plug in a known point on the line, e.g., $(5, 3)$:

$$5(5) - 6(3) = 7$$

Therefore, one equation is $\boxed{5x - 6y = 7}$. Other equations for this line are nonzero scalar multiples of this equation.

b) (7 pts) Which of the following are valid scalar-parametric forms of ℓ? **Select all that apply.** In each option, $t \in \mathbb{R}$.

☐ $\begin{cases} x = 5 + 12t \\ y = 4 + 10t \end{cases}$ ■ $\begin{cases} x = 11 + 12t \\ y = 8 + 10t \end{cases}$ ■ $\begin{cases} x = -1 - 18t \\ y = -2 - 15t \end{cases}$ ☐ $\begin{cases} x = 11 + 5t \\ y = 8 - 6t \end{cases}$

Solution: **Select the second and third options.** A parametrization describes ℓ when its starting point lies on $5x - 6y = 7$ and its nonzero direction is a scalar multiple of the original direction.

1. **Invalid:** The direction is correct, but $(5, 4)$ is not on ℓ: $5(5) - 6(4) = 1 \neq 7$.
2. **Valid:** $5(11) - 6(8) = 7$, and the direction is twice the original direction.
3. **Valid:** $5(-1) - 6(-2) = 7$, and the direction is -3 times the original direction.
4. **Invalid:** The starting point is on ℓ, but the direction is perpendicular to the original direction, rather than parallel to it.

c) (4 pts) Let ℓ' be the line perpendicular to ℓ that passes through $(7, -6)$. Write an equation for ℓ' in the form $ax + by = c$.

Solution: Because ℓ' is perpendicular to ℓ, the direction of ℓ is a normal to ℓ'. Thus ℓ' has an equation $6x + 5y = c$. Substituting its point $(7, -6)$ gives

$$c = 6(7) + 5(-6) = 12$$

One equation is $\boxed{6x + 5y = 12}$.

Problem 4 (11 pts)

Let $\vec{u}_1$ and $\vec{u}_2$ form an orthonormal basis of $\mathbb{R}^2$. Suppose

$$\vec{w} = 6\vec{u}_1 + 2\vec{u}_2.$$

a) (4 pts) Find $\|\vec{w}\|$.

Solution: Orthonormality gives $\|\vec{u}_1\| = \|\vec{u}_2\| = 1$ and $\vec{u}_1 \cdot \vec{u}_2 = 0$. Therefore,

$$\|\vec{w}\|^2 = (6\vec{u}_1 + 2\vec{u}_2) \cdot (6\vec{u}_1 + 2\vec{u}_2) = 36 + 24(0) + 4 = 40$$

So $\boxed{\|\vec{w}\| = \sqrt{40} = 2\sqrt{10}}$.

b) (3 pts) Find $\vec{w} \cdot \vec{u}_1$.

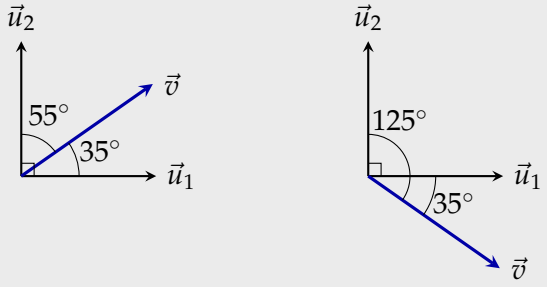
Solution: Distribute the dot product and use orthonormality:

$$\vec{w} \cdot \vec{u}_1 = 6(\vec{u}_1 \cdot \vec{u}_1) + 2(\vec{u}_2 \cdot \vec{u}_1) = 6(1) + 2(0) = \boxed{6}$$

c) (4 pts) Suppose $\vec{v}$ is another vector in $\mathbb{R}^2$. Given that the angle between $\vec{v}$ and $\vec{u}_1$ is $35°$, which of the following *could be* the angle between $\vec{v}$ and $\vec{u}_2$? **Select all that apply.**

☐ $35°$ ■ $55°$ ☐ $90°$ ■ $125°$ ☐ $145°$

Solution: The basis vectors $\vec{u}_1$ and $\vec{u}_2$ are perpendicular. We're given that the angle between $\vec{v}$ and $\vec{u}_1$ is $35°$—this could mean $\vec{v}$ is $35°$ "below" $\vec{u}_1$ or $35°$ "above" $\vec{u}_1$.



The angles with $\vec{u}_2$ are $90° - 35° = 55°$ and $90° + 35° = 125°$. These two pictures cover all possibilities up to rotation and reflection; the length of $\vec{v}$ does not affect its angles.

Problem 5 (12 pts)

Consider the vectors

$$\vec{u}_1 = \frac{1}{9}\begin{bmatrix}4\\4\\7\end{bmatrix}, \qquad \vec{u}_2 = \frac{1}{9}\begin{bmatrix}-8\\1\\4\end{bmatrix}, \qquad \vec{w} = \begin{bmatrix}4\\2\\3\end{bmatrix}$$

The vector $\vec{w}$ is in span$(\vec{u}_1, \vec{u}_2)$.

a) (6 pts) Verify that $\vec{u}_1$ and $\vec{u}_2$ are orthogonal and each has length 1.

Solution: Compute the dot product and both lengths:

$$\vec{u}_1 \cdot \vec{u}_2 = \frac{4(-8)+4(1)+7(4)}{81} = \frac{-32+4+28}{81} = 0$$

$$\|\vec{u}_1\| = \frac{\sqrt{4^2+4^2+7^2}}{9} = \frac{\sqrt{81}}{9} = 1$$

$$\|\vec{u}_2\| = \frac{\sqrt{(-8)^2+1^2+4^2}}{9} = \frac{\sqrt{81}}{9} = 1$$

The zero dot product verifies orthogonality, and the two lengths verify that both vectors are unit vectors.

b) (6 pts) Write $\vec{w}$ as a linear combination of $\vec{u}_1$ and $\vec{u}_2$.

Solution: Write $\vec{w} = a\vec{u}_1 + b\vec{u}_2$. Taking dot products with the orthonormal vectors isolates the coefficients:

$$a = \vec{w} \cdot \vec{u}_1 = \frac{4(4)+2(4)+3(7)}{9} = \frac{45}{9} = 5$$

$$b = \vec{w} \cdot \vec{u}_2 = \frac{4(-8)+2(1)+3(4)}{9} = \frac{-18}{9} = -2$$

Therefore, $\boxed{\vec{w} = 5\vec{u}_1 - 2\vec{u}_2}$. We can check the result directly:

$$5\vec{u}_1 - 2\vec{u}_2 = \frac{1}{9}\begin{bmatrix}20+16\\20-2\\35-8\end{bmatrix} = \begin{bmatrix}4\\2\\3\end{bmatrix} = \vec{w}$$

Problem 6 (16 pts)

Let P be the plane through $(6, -5, 8)$ with direction vectors

$$\vec{u} = \begin{bmatrix} 2 \\ 5 \\ 3 \end{bmatrix}, \qquad \vec{v} = \begin{bmatrix} 4 \\ 5 \\ 1 \end{bmatrix}$$

a) (8 pts) Write an equation for P in the form $ax + by + cz = d$.

Solution: Let $\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ be a normal. It must be perpendicular to both directions, so

$$\begin{cases} 2a + 5b + 3c = 0 \\ 4a + 5b + c = 0 \end{cases}$$

Subtract the first equation from the second to get $2a - 2c = 0$, or $c = a$. Substitute into the first:

$$2a + 5b + 3a = 0 \quad \Longrightarrow \quad b = -a$$

Choose $a = 1$, giving $\vec{n} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$. The plane therefore has an equation $x - y + z = d$. Its point $(6, -5, 8)$ determines the constant:

$$d = 6 - (-5) + 8 = 19$$

Thus $\boxed{P: \; x - y + z = 19}$. The two given directions are not scalar multiples, so they span a plane, and our normal is perpendicular to both.

b) (8 pts) For each vector-parametric equation below, select whether it is a valid description of P and give a brief justification. In every option, $s, t \in \mathbb{R}$.

Hint: Use the equation you found in the previous part.

i) (2 pts) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -5 \\ 8 \end{bmatrix} + s(\vec{u} + \vec{v}) + t\vec{v}$

● Valid ○ Invalid

Solution: **Valid.** The starting point is on P, and $\vec{u} + \vec{v}$ and $\vec{v}$ span the same directions as $\vec{u}$ and $\vec{v}$: recover $\vec{u} = (\vec{u} + \vec{v}) - \vec{v}$. Thus they still span the entire plane.

ii) (2 pts) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 9 \\ -4 \\ 6 \end{bmatrix} + s\vec{u} + t\vec{v}$

● Valid ○ Invalid

Solution: **Valid.** The new starting point satisfies $9 - (-4) + 6 = 19$, so it lies on P. The original two directions still span the plane; moving the starting point within P does not change the plane.

iii) (2 pts) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -5 \\ 9 \end{bmatrix} + s\vec{u} + t\vec{v}$

○ Valid ● Invalid

Solution: **Invalid.** The starting point does not satisfy the plane's equation:

$$6 - (-5) + 9 = 20 \neq 19$$

This parametrization describes a parallel plane, $x - y + z = 20$.

iv) (2 pts) $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -5 \\ 8 \end{bmatrix} + s(2\vec{u} - 3\vec{v}) + t(4\vec{u} - 6\vec{v})$

○ Valid ● Invalid

Solution: **Invalid.** The directions are scalar multiples:

$$4\vec{u} - 6\vec{v} = 2(2\vec{u} - 3\vec{v})$$

They provide only one independent direction, so this describes a line within P, rather than the whole plane.

Problem 7 (10 pts)

The equations

$$P_1 : \ 5x + 3y = 10, \qquad P_2 : \ 2x + 6y + 4z = 20$$

describe two planes in $\mathbb{R}^3$. Write the set of common solutions to these two equations in vector-parametric form.

Solution: **Method 1: Find a direction vector and a position vector.** The intersection line lies in both planes, so its direction vector must be parallel to both planes. Thus it must be perpendicular to both normal vectors,

$$\vec{n}_1 = \begin{bmatrix} 5 \\ 3 \\ 0 \end{bmatrix}, \qquad \vec{n}_2 = \begin{bmatrix} 2 \\ 6 \\ 4 \end{bmatrix}.$$

Write the direction vector as $\vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$. Requiring both dot products to be zero gives

$$\vec{v} \cdot \vec{n}_1 = 5a + 3b = 0, \qquad \vec{v} \cdot \vec{n}_2 = 2a + 6b + 4c = 0.$$

The first equation gives $b = -5a/3$. Substituting into the second,

$$2a + 6\left(-\frac{5a}{3}\right) + 4c = 0 \quad \Longrightarrow \quad -8a + 4c = 0 \quad \Longrightarrow \quad c = 2a.$$

Therefore $\vec{v} = a \begin{bmatrix} 1 \\ -5/3 \\ 2 \end{bmatrix}$. We need a nonzero direction vector; choosing $a = 3$ gives $b = -5$, $c = 6$, and $\vec{v} = \begin{bmatrix} 3 \\ -5 \\ 6 \end{bmatrix}$.

To find a position vector, find any one point that lies on *both* planes, so we know it is on their intersection line. Set $y = 0$. Then P_1 gives $5x = 10$, so $x = 2$, and P_2 gives $4 + 4z = 20$, so $z = 4$. Indeed,

$$5(2) + 3(0) = 10, \qquad 2(2) + 6(0) + 4(4) = 20.$$

Thus we can use $\begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}$ as a position vector, giving

$$\boxed{\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix} + t \begin{bmatrix} 3 \\ -5 \\ 6 \end{bmatrix}, \qquad t \in \mathbb{R}.}$$

The normals are not parallel, so the planes intersect in a line; this point and nonzero direction vector describe that entire line.

Solution: **Method 2: Solve directly for the common points.** First solve for y in terms of x using the first equation:

$$5x + 3y = 10 \quad \Longrightarrow \quad y = \frac{10 - 5x}{3}$$

Substitute this into the second equation to solve for z:

$$\begin{aligned} 2x + 6\left(\frac{10 - 5x}{3}\right) + 4z &= 20 \\ 2x + 20 - 10x + 4z &= 20 \\ z &= 2x \end{aligned}$$

Since x can be any real number, the common solutions are

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 10/3 \\ 0 \end{bmatrix} + x \begin{bmatrix} 1 \\ -5/3 \\ 2 \end{bmatrix}, \qquad x \in \mathbb{R}$$

This is already a valid answer. To avoid fractions, multiply the direction vector by 3, which does not change the line. Choosing the point corresponding to $x = 2$ as our starting point gives

$$\boxed{\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix} + t \begin{bmatrix} 3 \\ -5 \\ 6 \end{bmatrix}, \qquad t \in \mathbb{R}}$$

Equivalently, we have reparameterized by setting $x = 2 + 3t$. As t ranges over all real numbers, x does too, so we still describe the entire intersection line.

Problem 8 (12 pts)

Let

$$\vec{v} = \begin{bmatrix} 7 \\ 5 \\ 6 \end{bmatrix}, \qquad \ell = \text{span}\left(\begin{bmatrix} 4 \\ -5 \\ 7 \end{bmatrix} \right), \qquad P: \ 4x - 5y + 7z = 0$$

a) (6 pts) Find the orthogonal projection of $\vec{v}$ onto ℓ.

Solution: Let $\vec{w} = \begin{bmatrix} 4 \\ -5 \\ 7 \end{bmatrix}$, the vector spanning ℓ. Use the projection formula:

$$\begin{aligned} \text{proj}_\ell(\vec{v}) &= \frac{\vec{v} \cdot \vec{w}}{\vec{w} \cdot \vec{w}} \vec{w} \\ &= \frac{7(4) + 5(-5) + 6(7)}{4^2 + (-5)^2 + 7^2} \begin{bmatrix} 4 \\ -5 \\ 7 \end{bmatrix} \\ &= \frac{45}{90} \begin{bmatrix} 4 \\ -5 \\ 7 \end{bmatrix} = \boxed{\begin{bmatrix} 2 \\ -5/2 \\ 7/2 \end{bmatrix}} \end{aligned}$$

b) (6 pts) Find the orthogonal projection of $\vec{v}$ onto P.

Solution: The coefficients of P tell us that the vector $\begin{bmatrix} 4 \\ -5 \\ 7 \end{bmatrix}$ is perpendicular to P. But ℓ is the span of $\begin{bmatrix} 4 \\ -5 \\ 7 \end{bmatrix}$, so P and ℓ are perpendicular, and $\vec{v}$'s projections onto ℓ and onto P sum to $\vec{v}$ exactly. Thus,

$$\begin{aligned} \text{proj}_P(\vec{v}) &= \vec{v} - \text{proj}_\ell(\vec{v}) \\ &= \begin{bmatrix} 7 \\ 5 \\ 6 \end{bmatrix} - \begin{bmatrix} 2 \\ -5/2 \\ 7/2 \end{bmatrix} = \boxed{\begin{bmatrix} 5 \\ 15/2 \\ 5/2 \end{bmatrix}} \end{aligned}$$

As a check, this vector lies in P:

$$4(5) - 5\left(\frac{15}{2}\right) + 7\left(\frac{5}{2}\right) = 0$$