# Midterm 2 Solutions

Math 124, Fall 2026

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**Room:** ◯ 0460 CCCB ◯ 2866 East Hall ◯ 1866 East Hall

**Instructions**

- This exam consists of 8 problems across 13 pages, worth a total of 100 points.
- You have 120 minutes to complete this exam, unless you have extended-time accommodations through SSD.
- Write your uniqname in the top right corner of each page where a space is provided.
- For free response problems, **you must show all of your work**, and write your final answer in the small box when one is provided.
- For multiple choice problems, completely fill in bubbles and square boxes.
  ◯ A bubble means that you should only select one choice.
  ☐ A square box means that you should select all that apply.
- You may refer to **two double-sided** $4 \times 6$**-inch index cards**, handwritten directly on the cards. Other than that, you may not refer to any other resources or technology during the exam (no phones, watches, or calculators).

By signing below, I affirm that this work is my own and that I have followed the exam's resource and collaboration rules.

**Problem 1 (8 pts)**

**a)** (4 pts) Let

$$C = \begin{bmatrix} 4 & -2 \\ -3 & 5 \\ 6 & 1 \end{bmatrix}, \qquad D = \begin{bmatrix} 2 & -4 & 3 \\ 5 & 1 & -2 \\ -3 & 6 & 4 \\ 7 & -2 & 5 \end{bmatrix}.$$

Give the shape of each expression (e.g. $5 \times 3$), or write "undefined."

$$CD \qquad DC \qquad C + C \qquad D(3C).$$

**Solution:** The shape of $C$ is $3 \times 2$, and the shape of $D$ is $4 \times 3$. For a product, the inner dimensions must agree, and the outer dimensions give its shape. Thus

$$\boxed{CD: \text{ undefined}, \qquad DC: \ 4 \times 2, \qquad C + C: \ 3 \times 2, \qquad D(3C): \ 4 \times 2.}$$

The product $CD$ is undefined because $2 \neq 4$. Adding $C$ to itself or multiplying it by 3 does not change its shape.

**b)** (4 pts) Let $a$ be a real constant, and let

$$A = \begin{bmatrix} a & 4 \\ 5 & a \\ -1 & -7 \end{bmatrix}, \qquad B = \begin{bmatrix} 3 & 4 & 2 \\ -2 & 5 & -1 \end{bmatrix}.$$

Suppose the third column of $AB$ is

$$\begin{bmatrix} 2 \\ m \\ 5 \end{bmatrix}.$$

**i)** (2 pts) Find the value of $a$. Show your work.

**Solution:** The third column of $AB$ is $A$ times the third column of $B$:

$$A \begin{bmatrix} 2 \\ -1 \end{bmatrix} = 2 \begin{bmatrix} a \\ 5 \\ -1 \end{bmatrix} - \begin{bmatrix} 4 \\ a \\ -7 \end{bmatrix} = \begin{bmatrix} 2a - 4 \\ 10 - a \\ 5 \end{bmatrix}.$$

Matching the first entry gives $2a - 4 = 2$, so $\boxed{a = 3}$.

**ii)** (2 pts) Find the value of $m$. Show your work.

**Solution:** The second entry of the third column is $10 - a$. Using $a = 3$ from part (i), we get

$$\boxed{m = 10 - 3 = 7.}$$

**Problem 2 (12 pts)**

**a)** (8 pts) Suppose a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ satisfies

$$T\left(\begin{bmatrix} 2 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ 2 \\ -3 \end{bmatrix}, \quad T\left(\begin{bmatrix} -1 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} -2 \\ 5 \\ 1 \end{bmatrix}.$$

**i)** (1 pt) What is the shape of the matrix $A$ where $T(\vec{x}) = A\vec{x}$?

**Solution:** The matrix takes a vector with 2 entries as input and produces a vector with 3 entries. It therefore has 3 rows and 2 columns, so its shape is $\boxed{3 \times 2}$.

**ii)** (7 pts) Find an input vector $\vec{v} \in \mathbb{R}^2$ such that

$$T(\vec{v}) = \begin{bmatrix} 10 \\ -1 \\ -7 \end{bmatrix}.$$

**Solution:** We want to write the desired output as a linear combination of the two outputs we know. Suppose

$$\alpha \begin{bmatrix} 4 \\ 2 \\ -3 \end{bmatrix} + \beta \begin{bmatrix} -2 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ -1 \\ -7 \end{bmatrix}.$$

The first two entries give $4\alpha - 2\beta = 10$ and $2\alpha + 5\beta = -1$. From the first equation, $\beta = 2\alpha - 5$. Substituting into the second gives $12\alpha = 24$, so $\alpha = 2$ and $\beta = -1$. These values also give the required third entry, since $2(-3) - 1 = -7$.
By linearity, we can use the same coefficients on the inputs:

$$T\left(2\begin{bmatrix} 2 \\ -1 \end{bmatrix} - \begin{bmatrix} -1 \\ 3 \end{bmatrix}\right) = 2\begin{bmatrix} 4 \\ 2 \\ -3 \end{bmatrix} - \begin{bmatrix} -2 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ -1 \\ -7 \end{bmatrix}.$$

Therefore, $\boxed{\vec{v} = \begin{bmatrix} 5 \\ -5 \end{bmatrix}}$.

**b)** (4 pts) Which of the following transformations from $\mathbb{R}^2$ to $\mathbb{R}^2$ are linear? **Select all that apply.**

☐ Multiply each vector by its length.

■ Swap the horizontal and vertical components.

☐ Add 4 to the horizontal component and subtract 2 from the vertical component.

■ Send every vector to the zero vector.

**Solution:** Select **the second and fourth choices**.

- Multiplying each vector by its length is not linear. The vector $\begin{bmatrix}1\\0\end{bmatrix}$ maps to itself, but twice that vector, $\begin{bmatrix}2\\0\end{bmatrix}$, maps to $\begin{bmatrix}4\\0\end{bmatrix}$. Doubling the input does not double the output.
- Swapping the components is linear: it is multiplication by $\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$.
- Adding 4 to one component and subtracting 2 from the other is not linear. It sends $\vec{0}$ to $\begin{bmatrix}4\\-2\end{bmatrix}$, but a linear transformation must send $\vec{0}$ to $\vec{0}$.
- Sending every vector to $\vec{0}$ is linear: it is multiplication by the zero matrix.

**Problem 3 (12 pts)**

**a)** (4 pts) Let

$$A = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}.$$

This matrix projects vectors orthogonally onto a line $\ell$ through the origin. Give a nonzero vector $\vec{u}$ such that $\ell = \text{span}(\vec{u})$.

**Solution:** The columns of $A$ are the projections of the standard basis vectors, so both lie on $\ell$. The first column is the nonzero vector $\frac{1}{5}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, which gives the direction of the line. One answer is therefore

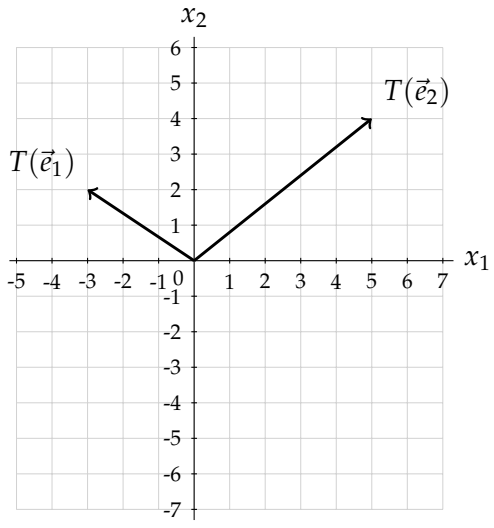
$$\boxed{\vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.}$$

Any nonzero scalar multiple of this vector also works.

**b)** (8 pts) The diagram shows $T(\vec{e}_1)$ and $T(\vec{e}_2)$ for a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$, where the standard basis vectors are

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The transformation $S$ first applies $T$, **then** rotates the resulting vector 90° **clockwise** about the origin. Find the $2 \times 2$ matrix $B$ corresponding to $S$ (**not** $T$).




**Solution:** From the diagram,

$$T(\vec{e}_1) = \begin{bmatrix} -3 \\ 2 \end{bmatrix}, \qquad T(\vec{e}_2) = \begin{bmatrix} 5 \\ 4 \end{bmatrix}.$$

A 90° clockwise rotation sends $\begin{bmatrix} r \\ s \end{bmatrix}$ to $\begin{bmatrix} s \\ -r \end{bmatrix}$. Therefore,

$$S(\vec{e}_1) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \qquad S(\vec{e}_2) = \begin{bmatrix} 4 \\ -5 \end{bmatrix}.$$

These outputs are the first and second columns of the matrix for $S$, so

$$\boxed{B = \begin{bmatrix} 2 & 4 \\ 3 & -5 \end{bmatrix}.}$$

**Problem 4 (12 pts)**

**a)** (5 pts) Find $A^{-1}$, where

$$A = \begin{bmatrix} 7 & -3 \\ 5 & -1 \end{bmatrix}.$$

**Solution:** The $2 \times 2$ inverse formula gives

$$A^{-1} = \frac{1}{7(-1) - (-3)(5)} \begin{bmatrix} -1 & 3 \\ -5 & 7 \end{bmatrix} = \boxed{\frac{1}{8} \begin{bmatrix} -1 & 3 \\ -5 & 7 \end{bmatrix}}.$$

The denominator is $-7 + 15 = 8 \neq 0$, so the inverse exists.

**b)** (7 pts) Two invertible matrices $C$ and $D$ have inverses

$$C^{-1} = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}, \qquad D^{-1} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}.$$

A vector $\vec{x}$ is first multiplied by $C$, and the result is then multiplied by $D$. The final output is

$$\begin{bmatrix} 5 \\ 2 \end{bmatrix}.$$

Find $\vec{x}$.

**Solution:** The given output is $DC\vec{x}$. We need to undo $D$ first, then undo $C$. Multiplying the output by $D^{-1}$ gives

$$C\vec{x} = D^{-1} \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Now multiply by $C^{-1}$:

$$\vec{x} = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \boxed{\begin{bmatrix} 8 \\ 7 \end{bmatrix}}.$$

**Problem 5 (10 pts)**

Find matrices $L$ and $U$ such that $A = LU$, where

$$A = \begin{bmatrix} 5 & 2 & -4 \\ 15 & 3 & -5 \\ -10 & -1 & 7 \end{bmatrix}.$$

Here, $L$ must be lower triangular with ones on its diagonal, and $U$ must be upper triangular. No row swaps are necessary.

**Solution:** Write the factors in the required forms:

$$L = \begin{bmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{bmatrix}, \qquad U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}.$$

The first row of $LU$ is the first row of $U$, so $u_{11} = 5$, $u_{12} = 2$, and $u_{13} = -4$. Matching the first column gives

$$5\ell_{21} = 15, \qquad 5\ell_{31} = -10, \qquad \text{so} \qquad \ell_{21} = 3, \quad \ell_{31} = -2.$$

Matching the remaining entries in row 2 gives

$$3(2) + u_{22} = 3, \qquad 3(-4) + u_{23} = -5,$$

so $u_{22} = -3$ and $u_{23} = 7$. Entry $(3, 2)$ then gives

$$(-2)(2) + \ell_{32}(-3) = -1, \qquad \text{so} \qquad \ell_{32} = -1.$$

Finally, entry $(3, 3)$ gives $(-2)(-4) + (-1)(7) + u_{33} = 7$, so $u_{33} = 6$. Thus

$$\boxed{L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -2 & -1 & 1 \end{bmatrix}, \qquad U = \begin{bmatrix} 5 & 2 & -4 \\ 0 & -3 & 7 \\ 0 & 0 & 6 \end{bmatrix}.}$$

Multiplying these factors gives $A$, as required.

**Problem 6 (18 pts)**

**a)** (10 pts) Suppose a $3 \times 5$ matrix $A$ has the factorization $PA = LU$, where

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad L = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & -1 & 1 \end{bmatrix}, \quad U = \begin{bmatrix} 4 & -8 & 3 & 2 & -5 \\ 0 & 0 & 5 & -10 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Use the given factorization to find all real values of $c$ for which the following system has no solutions:

$$A\vec{x} = \begin{bmatrix} c \\ 4 \\ 7 \end{bmatrix}.$$

**Solution:** Let $\vec{b} = \begin{bmatrix} c \\ 4 \\ 7 \end{bmatrix}$. Since $PA = LU$, we can solve $L\vec{y} = P\vec{b}$ first, then consider $U\vec{x} = \vec{y}$. The permutation changes the order of the entries:

$$P\vec{b} = \begin{bmatrix} 4 \\ 7 \\ c \end{bmatrix}.$$

Forward substitution in $L\vec{y} = P\vec{b}$ gives

$$\begin{aligned} y_1 &= 4, \\ -2y_1 + y_2 = 7 \quad &\Longrightarrow \quad y_2 = 15, \\ 3y_1 - y_2 + y_3 = c \quad &\Longrightarrow \quad y_3 = c + 3. \end{aligned}$$

The last row of $U\vec{x} = \vec{y}$ is therefore $0 = c + 3$. If $c \neq -3$, this is impossible, so there are no solutions.

If $c = -3$, the last equation is $0 = 0$. The second equation determines $x_3$ after choosing $x_4, x_5$, and the first determines $x_1$ after choosing $x_2$. Thus the system is consistent when $c = -3$, and the requested values are exactly

$$\boxed{c \neq -3.}$$

**b)** (8 pts) Find all solutions of the following system. Give your answer in vector-parametric form.

$$\begin{bmatrix} 5 & -4 & 3 \\ 0 & 6 & -12 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ -18 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

**Solution:** The only nonzero equations are

$$5x_1 - 4x_2 + 3x_3 = 7, \qquad 6x_2 - 12x_3 = -18.$$

The last three rows give $0 = 0$, so they add no restrictions. Let the free variable be $x_3 = t$. The second equation gives

$$x_2 = -3 + 2t.$$

Substituting into the first gives $5x_1 - 4(-3 + 2t) + 3t = 7$, so $5x_1 = -5 + 5t$ and $x_1 = -1 + t$. Therefore all solutions are

$$\boxed{\vec{x} = \begin{bmatrix} -1 \\ -3 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \qquad t \in \mathbb{R}.}$$

**Problem 7 (11 pts)**

**a)** (8 pts) Let $S$ be the plane $3x + 2y - z = 0$ in $\mathbb{R}^3$. Which of the following matrices $A$ satisfy $\text{colsp}(A) = S$? **Select all that apply.**

■ $A = \begin{bmatrix} 2 & -1 \\ -1 & 3 \\ 4 & 3 \end{bmatrix}$

■ $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 3 & 2 \\ 4 & 3 & 7 \end{bmatrix}$

☐ $A = \begin{bmatrix} 2 & 6 \\ -1 & -3 \\ 4 & 12 \end{bmatrix}$

☐ $A = \begin{bmatrix} 2 & -1 \\ -1 & 3 \\ 4 & 5 \end{bmatrix}$

**Solution: Select the first and second matrices.** Let

$$\vec{u} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}, \qquad \vec{v} = \begin{bmatrix} -1 \\ 3 \\ 3 \end{bmatrix}.$$

Both vectors belong to $S$, since

$$3(2) + 2(-1) - 4 = 0, \qquad 3(-1) + 2(3) - 3 = 0.$$

They are also linearly independent. If $\vec{v} = k\vec{u}$, the first entries would require $k = -\frac{1}{2}$, but then the second entry of $k\vec{u}$ would be $\frac{1}{2}$, not 3. Thus they are not scalar multiples of one another.

Since $S$ is a plane through the origin, it has dimension 2. Two linearly independent vectors in $S$ form a basis for $S$, so the first matrix has column space $S$.

The second matrix has the same first two columns, and its third column is their sum:

$$\begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix} = \vec{u} + \vec{v}.$$

This extra column does not change the column space, so the second matrix also works.

The third matrix does not work: its second column is $3\vec{u}$, so its column space is only the line spanned by $\vec{u}$.

The fourth matrix does not work either. Its second column is not even in $S$, since

$$3(-1) + 2(3) - 5 = -2 \neq 0.$$

Therefore the correct choices are the first and second matrices.

**b)** (3 pts) Recall that a subspace contains the zero vector and is closed under addition and scalar multiplication. Consider the set

$$W = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 : xy = 0 \right\}.$$

This set is **not** a subspace of $\mathbb{R}^2$, but it still satisfies some of these properties. For each property below, select **yes or no**.

**i)** (1 pt) Does $W$ contain the zero vector?

● Yes ○ No

**ii)** (1 pt) Is $W$ closed under addition?

○ Yes ● No

**iii)** (1 pt) Is $W$ closed under scalar multiplication by any real number?

● Yes ○ No

**Solution:** **i)** Yes. The zero vector belongs to $W$, since $0 \cdot 0 = 0$.

**ii)** No. For example, both $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ belong to $W$, but their sum does not:

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad 1 \cdot 1 \neq 0.$$

**iii)** Yes. If $\begin{bmatrix} x \\ y \end{bmatrix} \in W$, then $xy = 0$. For any real scalar $a$, the product of the entries of $a \begin{bmatrix} x \\ y \end{bmatrix}$ is

$$(ax)(ay) = a^2xy = 0.$$

Thus the scalar multiple also belongs to $W$.

**Problem 8 (17 pts)**

**a)** (6 pts) Let $\vec{x}, \vec{y}, \vec{z} \in \mathbb{R}^5$ be linearly independent, and let $t \in \mathbb{R}$. Define

$$A = \begin{bmatrix} | & | \\ 2\vec{x} - \vec{y} & 6\vec{x} + t\vec{y} \\ | & | \end{bmatrix}, \qquad B = \begin{bmatrix} | & | & | \\ \vec{x} + \vec{y} & \vec{y} - 4\vec{z} & \vec{x} + t\vec{z} \\ | & | & | \end{bmatrix}.$$

For each part below, select the **one value of** $t$ that satisfies the given condition.

**i)** (3 pts) $\text{rank}(A) = 1$

**Solution:** The first column, $2\vec{x} - \vec{y}$, is nonzero: otherwise $\vec{y} = 2\vec{x}$, contradicting linear independence. Therefore $A$ has rank 1 exactly when its second column is a scalar multiple of its first. Write

$$6\vec{x} + t\vec{y} = k(2\vec{x} - \vec{y}) = 2k\vec{x} - k\vec{y}.$$

Because $\vec{x}$ and $\vec{y}$ are linearly independent, their coefficients on the two sides must match. This gives $6 = 2k$ and $t = -k$, so

$$\boxed{t = -3}.$$

For this value, the second column is three times the first, as required.

**ii)** (3 pts) $\text{rank}(B) = 2$

**Solution:** Try to write the first column as a scalar multiple of the second:

$$\vec{x} + \vec{y} = k(\vec{y} - 4\vec{z}) \quad \Longrightarrow \quad \vec{x} = (k-1)\vec{y} - 4k\vec{z}.$$

This is impossible: it writes $\vec{x}$ as a linear combination of $\vec{y}$ and $\vec{z}$. The second column is also nonzero, since otherwise $\vec{y} = 4\vec{z}$. Thus the first two columns are linearly independent.

Thus $B$ has rank 2 exactly when its third column is a linear combination of the first two. Suppose

$$\vec{x} + t\vec{z} = a(\vec{x} + \vec{y}) + b(\vec{y} - 4\vec{z}) = a\vec{x} + (a+b)\vec{y} - 4b\vec{z}.$$

Matching the coefficients of $\vec{x}, \vec{y}, \vec{z}$ gives

$$a = 1, \qquad a + b = 0, \qquad t = -4b.$$

So $b = -1$ and $\boxed{t = 4}$. For this value, the third column is the first minus the second:

$$\vec{x} + 4\vec{z} = (\vec{x} + \vec{y}) - (\vec{y} - 4\vec{z}).$$

**b)** (2 pts) A matrix $D$ has rank 8. The solutions of $D\vec{x} = \vec{0}$ have 5 free variables. How many columns does $D$ have?

**Solution:** Rank 8 means there are 8 pivot columns. Each of the 5 free variables corresponds to a non-pivot column. Thus the total number of columns is

$$\boxed{8 + 5 = 13}.$$

**c)** (6 pts) Suppose $E$ is a $3 \times 3$ matrix such that

$$E \begin{bmatrix} 8 \\ -5 \\ 11 \end{bmatrix} = \begin{bmatrix} 6 \\ -8 \\ 13 \end{bmatrix}, \qquad E \begin{bmatrix} -3 \\ 7 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

**i)** (2 pts) True or false: $E$ is invertible.

◯ True ● False

**Solution:** False. The nonzero vector $\begin{bmatrix} -3 \\ 7 \\ 4 \end{bmatrix}$ is mapped to the zero vector. If $E$ were invertible, multiplying both sides of the given equality by $E^{-1}$ would give

$$\begin{bmatrix} -3 \\ 7 \\ 4 \end{bmatrix} = E^{-1} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

This is impossible. Thus $E$ is not invertible.

**ii)** (4 pts) Give a solution of the following system **other than** $\begin{bmatrix} 8 \\ -5 \\ 11 \end{bmatrix}$:

$$E\vec{x} = \begin{bmatrix} 6 \\ -8 \\ 13 \end{bmatrix}.$$

**Solution:** We can add a vector that $E$ sends to zero to the given solution without changing the output. In particular,

$$E\left( \begin{bmatrix} 8 \\ -5 \\ 11 \end{bmatrix} + \begin{bmatrix} -3 \\ 7 \\ 4 \end{bmatrix} \right) = E \begin{bmatrix} 8 \\ -5 \\ 11 \end{bmatrix} + E \begin{bmatrix} -3 \\ 7 \\ 4 \end{bmatrix} = \begin{bmatrix} 6 \\ -8 \\ 13 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Therefore one valid answer is

$$\boxed{\vec{x} = \begin{bmatrix} 5 \\ 2 \\ 15 \end{bmatrix}}.$$

**d)** (3 pts) A $6 \times 4$ matrix $G$ has rank 4. Is $G\vec{x} = \vec{b}$ guaranteed to have a solution for every $\vec{b} \in \mathbb{R}^6$? Briefly explain.

**Solution:** No. The column space of $G$ has dimension 4, while $\mathbb{R}^6$ has dimension 6. Thus the columns of $G$ cannot span all of $\mathbb{R}^6$. If $\vec{b}$ lies outside the column space, $G\vec{x} = \vec{b}$ has no solution.