Homework 7: Linear Systems

Math 124, Fall 2026 at the University of Michigan

due Sunday, November 8th at 11:59PM

Write your solutions to the following problems either by writing them on a piece of paper or on a tablet and scanning your answers as a PDF. Note that you are not allowed to use LaTeX, Google Docs, or any other digital document creation software to type your answers. Homeworks are due to Pensive by 11:59PM on the due date. See the syllabus for details on the slip day policy.

Homework will be evaluated not only on the correctness of your answers, but on your ability to present your ideas clearly and logically. You should always explain and justify your conclusions, using sound reasoning. Your goal should be to convince the reader of your assertions. If a question does not require explanation, it will be explicitly stated.

Before proceeding, make sure you're familiar with the collaboration policy.

Total Points: $6 + 15 + 20 + 15 + 26 + 18 = 100$

Problem 1: Homework 6 Solutions Review (6 pts)

Review the solutions to Homework 6. Pick **two problem parts** (for example, Problem 3a and Problem 5b) from Homework 6 in which your solutions have the most room for improvement, i.e., where they have unsound reasoning, could be significantly more efficient or clearer, etc. **Include a screenshot of your solution to each problem part**, and in a few sentences, explain what was deficient and how it could be fixed.

Alternatively, if you think one of your solutions is significantly better than the posted one, copy it here and explain why you think it is better. If you didn't do Homework 6, choose two problem parts from it that look challenging to you, and in a few sentences, explain the key ideas behind their solutions in your own words.

Problem 2: When a Pivot Goes Missing (15 pts)

Let

$$A = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & 0 \\ 3 & -6 & 1 \end{bmatrix}.$$

a) (7 pts) Find a factorization $A = LU$, where L is lower triangular with ones on its diagonal and U is in row echelon form. Identify the pivot columns of U.
Hint: Don't forget about the requirement that U is in row echelon form!

b) (8 pts) Use your factors to find all solutions of each system:

$$A\vec{x} = \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix}, \qquad A\vec{x} = \begin{bmatrix} 3 \\ -2 \\ 6 \end{bmatrix}.$$

If there are infinitely many solutions, give a vector-parametric description of the solution set. If there is no solution, explain why.

Problem 3: Factoring a Rectangular Matrix (20 pts)

Let

$$B = \begin{bmatrix} 2 & -2 & 1 & 3 \\ 6 & -6 & 3 & 9 \\ -2 & 2 & 3 & 1 \end{bmatrix}.$$

a) (5 pts) Attempt to find a factorization $B = LU$ without reordering the rows, where L has ones on its diagonal and U is in row echelon form. Explain why a row exchange is necessary.

b) (8 pts) Find a permutation matrix P and factors L, U such that $PB = LU$, with L square and lower triangular with ones on its diagonal and U in row echelon form. Identify the pivot columns of U, and verify your factorization by multiplying LU and showing it is equal to PB.

c) (7 pts) Use your factors to find all solutions of

$$B\vec{x} = \begin{bmatrix} 6 \\ 18 \\ 2 \end{bmatrix}.$$

Give a vector-parametric description of the solution set.

Problem 4: Working Backward (15 pts)

For each part, give a matrix U in row echelon form and a vector $\vec{y}$ such that the system $U\vec{x} = \vec{y}$ has the stated properties. Write your system as a matrix equation, and give a vector-parametric description of the solution set, or explain why it has no solution. Each part may use a different matrix.

a) (3 pts) U is 3×3, and the system has a unique solution.

b) (3 pts) U is 3×3, and the solution set is a line in $\mathbb{R}^3$ that does not pass through the origin.

c) (3 pts) U is 3×4, and the solution set is a plane through the origin in $\mathbb{R}^4$.

d) (3 pts) U is 4×3, and the system has no solution.

e) (3 pts) U is 4×2, and the system has a unique solution.

Problem 5: A Plane of Possibilities (26 pts)

Suppose $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, and let C be the 3×2 matrix whose columns are $\vec{v}$ and $\vec{w}$:

$$C = \begin{bmatrix} | & | \\ \vec{v} & \vec{w} \\ | & | \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}.$$

A possible output of a matrix is any vector that can be produced by multiplying that matrix by a vector. For example, $\begin{bmatrix} 5 \\ 9 \\ 13 \end{bmatrix}$ is a possible output of C, since

$$\begin{bmatrix} 5 \\ 9 \\ 13 \end{bmatrix} = C \begin{bmatrix} 1 \\ 4 \end{bmatrix}.$$

a) (4 pts) Find a factorization $C = LU$, where L is square and lower triangular with ones on its diagonal and U is in row echelon form.

b) (6 pts) What are all possible outputs of C? Using your factorization $C = LU$, determine the conditions on x, y, and z such that

$$C\vec{u} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

has a solution $\vec{u}$. The condition you find should be a single equation involving x, y, and z.

c) (4 pts) Under the condition you found in part b), find the solution for the vector $\vec{u} \in \mathbb{R}^2$ in terms of x, y, and z. Is the solution unique?

d) (8 pts) Using your understanding of geometry and span from earlier in the semester, find the equation of the plane spanned by $\vec{v}$ and $\vec{w}$. Why do you get the same answer as in part b)?

e) (4 pts) Consider the matrix

$$F = \begin{bmatrix} | & | & | \\ \vec{v} & \vec{w} & \vec{v}+\vec{w} \\ | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \end{bmatrix}.$$

Does F have the same possible outputs as C? Explain. Then, find two different inputs that F sends to

$$2\vec{v} + 3\vec{w} = \begin{bmatrix} 5 \\ 8 \\ 11 \end{bmatrix}.$$

Problem 6: The Score Adds Up (18 pts)

Before proceeding, open the supplemental Jupyter Notebook. To open the notebook for Homework 7, click **this link**. Instructions on how to use Google Colab are at math124.org/running-code.

Complete Problem 6 in the notebook. Tasks 1–8 are worth $1 + 1 + 2 + 5 + 1 + 2 + 1 + 5 = 18$ points, respectively. Follow its orange instructions for what to include in your PDF submission to Pensive.