Homework 8: Subspaces, Bases, and Rank Solutions

Math 124, Fall 2026 at the University of Michigan

due Friday, November 13th at 11:59PM

Write your solutions to the following problems either by writing them on a piece of paper or on a tablet and scanning your answers as a PDF. Note that you are not allowed to use LaTeX, Google Docs, or any other digital document creation software to type your answers. Homeworks are due to Pensive by 11:59PM on the due date. See the syllabus for details on the slip day policy.

Homework will be evaluated not only on the correctness of your answers, but on your ability to present your ideas clearly and logically. You should always explain and justify your conclusions, using sound reasoning. Your goal should be to convince the reader of your assertions. If a question does not require explanation, it will be explicitly stated.

Before proceeding, make sure you're familiar with the collaboration policy.

Total Points: $6 + 8 + 21 + 10 + 7 + 21 + 27 = 100$.

Note: Refer to **Chapter 5** while working on the homework!

Problem 1: Homework 7 Solutions Review (6 pts)

Review the solutions to Homework 7. Pick **two problem parts** (for example, Problem 3a and Problem 5b) from Homework 7 in which your solutions have the most room for improvement, i.e., where they have unsound reasoning, could be significantly more efficient or clearer, etc. **Include a screenshot of your solution to each problem part**, and in a few sentences, explain what was deficient and how it could be fixed.

Alternatively, if you think one of your solutions is significantly better than the posted one, copy it here and explain why you think it is better. If you didn't do Homework 7, choose two problem parts from it that look challenging to you, and in a few sentences, explain the key ideas behind their solutions in your own words.

Solution: Responses will vary. Identify the specific issue in each selected solution and explain how to improve the reasoning or presentation.

Problem 2: Multiplying a column by a row (8 pts)

Let $A = \begin{bmatrix} 5 \\ 1 \\ 3 \\ -1 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 0 & 10 & -2 & 1 \end{bmatrix}$.

a) (4 pts) Write AB in matrix form. *Hint: The result is a* 4×5 *matrix.*

Solution: Each entry of A multiplies the entire row B:

$$AB = \begin{bmatrix} 5(4) & 5(0) & 5(10) & 5(-2) & 5(1) \\ 1(4) & 1(0) & 1(10) & 1(-2) & 1(1) \\ 3(4) & 3(0) & 3(10) & 3(-2) & 3(1) \\ -1(4) & -1(0) & -1(10) & -1(-2) & -1(1) \end{bmatrix} = \boxed{\begin{bmatrix} 20 & 0 & 50 & -10 & 5 \\ 4 & 0 & 10 & -2 & 1 \\ 12 & 0 & 30 & -6 & 3 \\ -4 & 0 & -10 & 2 & -1 \end{bmatrix}}.$$

b) (2 pts) Fill in the blanks: Every column of AB is a scalar multiple of _____, while every row of AB is a scalar multiple of _____.

Solution: The blanks are

$$\boxed{\begin{bmatrix} 5 \\ 1 \\ 3 \\ -1 \end{bmatrix}} \quad \text{and} \quad \boxed{\begin{bmatrix} 4 & 0 & 10 & -2 & 1 \end{bmatrix}}.$$

The columns are $4A, 0A, 10A, -2A, A$. The rows are $5B, B, 3B, -B$.

c) (2 pts) Find $\text{rank}(AB)$.

Solution: $\boxed{\text{rank}(AB) = 1}$. Every column is a multiple of A, and A itself is the last column. Thus

$$\text{colsp}(AB) = \text{span}\left(\begin{bmatrix} 5 \\ 1 \\ 3 \\ -1 \end{bmatrix} \right).$$

This nonzero vector is a basis of the column space, so its dimension is 1. The zero column does not make the rank 0.

Problem 3: Column spaces and rank (21 pts)

Consider the following matrices.

$$A = \begin{bmatrix} 4 & -2 \\ 10 & 0 \\ 1 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 3 & -4 \\ 4 & 3 \\ 0 & 0 \end{bmatrix}, \qquad C = \begin{bmatrix} 6 & 2 \\ 10 & 10 \\ 0 & 2 \end{bmatrix}.$$

$$D = \begin{bmatrix} 4 & -2 & 2 \\ 10 & 0 & 10 \\ 1 & 1 & 2 \end{bmatrix}, \quad E = \begin{bmatrix} 3 & -4 & 0 \\ 4 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}, \quad F = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 5 & 10 & -5 & 15 \\ 1 & 2 & -1 & 3 \end{bmatrix}, \quad G = \begin{bmatrix} 4 & -2 & 6 & -4 \\ 10 & 0 & 10 & 0 \\ 1 & 1 & 0 & 2 \end{bmatrix}.$$

a) (14 pts) For each matrix, determine whether its column space is exactly the plane $5x - 3y + 10z = 0$ in $\mathbb{R}^3$.

Solution: Call the plane P. To have column space exactly P, all columns must lie on P, and they must span the whole plane. Having all columns on P is not enough if their span is only a line. Let

$$\vec{u} = \begin{bmatrix} 4 \\ 10 \\ 1 \end{bmatrix}, \qquad \vec{v} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}.$$

Both are on P, since $5(4) - 3(10) + 10(1) = 0$ and $5(-2) - 3(0) + 10(1) = 0$. They are not scalar multiples, so they span a plane. Since that plane is contained in P, their span is exactly P.

- A**: Yes.** Its columns are $\vec{u}, \vec{v}$, which span P.
- B**: No.** Its first column is not on P: $5(3) - 3(4) + 10(0) = 3 \neq 0$. This column is itself an output of B, so the column space includes a vector outside P.
- C**: Yes.** Its columns are $\vec{u} - \vec{v}$ and $\vec{u} + \vec{v}$. They lie on P, and adding and subtracting them recovers $2\vec{u}$ and $2\vec{v}$. Thus they span all of P.
- D**: Yes.** Its columns are $\vec{u}, \vec{v}, \vec{u} + \vec{v}$. The third column adds no new outputs, so its column space is still P.
- E**: No.** As with B, its first column is not on P. Its column space is actually all of $\mathbb{R}^3$.
- F**: No.** Its columns are $\vec{f}, 2\vec{f}, -\vec{f}, 3\vec{f}$, where $\vec{f} = \begin{bmatrix} 1 \\ 5 \\ 1 \end{bmatrix}$ lies on P. They span only the line through $\vec{f}$. For example, $\vec{u}$ lies on P but is not a multiple of $\vec{f}$, so it is not an output of F.
- G**: Yes.** Its columns are $\vec{u}, \vec{v}, \vec{u} - \vec{v}, 2\vec{v}$. The last 2 columns are redundant, so its column space is P.

The matrices with the requested column space are $\boxed{A, C, D, G}$.

b) (7 pts) Find the rank of each matrix.

Solution: Rank is the dimension of the column space, or the maximum number of linearly independent columns.

Matrix	Rank	Reason
A	2	Its 2 columns are independent and span P.
B	2	Its 2 nonzero columns are orthogonal: $3(-4)+4(3)+0(0)=0$. They are independent and span the plane $z=0$.
C	2	Its 2 columns span P and are independent.
D	2	The first 2 columns form a basis; the third is their sum.
E	3	The first 2 columns span $z=0$; the third, $\begin{bmatrix}0\\0\\6\end{bmatrix}$, lies outside that plane. All 3 columns are independent.
F	1	All columns are multiples of the same nonzero vector.
G	2	The first 2 columns form a basis; the remaining columns are redundant.

In particular, A and B have the same rank but different column spaces. Rank tells us the dimension of the output set, not which plane it is.

Problem 4: Reachable outputs and unique inputs (10 pts)

For parts (a)–(c), suppose A has shape 5×7 and rank 5. Determine whether each statement is **true or false**, and explain your answer.

a) (2 pts) $\text{colsp}(A) = \mathbb{R}^5$.

> **Solution:** **True.** A has 5 rows, so its column space is a subspace of $\mathbb{R}^5$. Its rank is 5, so that subspace has dimension 5 and must be all of $\mathbb{R}^5$.

b) (2 pts) The columns of A are linearly independent.

> **Solution:** **False.** There are 7 columns, but the maximum number of independent columns is the rank, 5. Therefore the full collection of 7 columns is dependent.

c) (2 pts) There is exactly one input $\vec{x}$ such that $A\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$.

> **Solution:** **False.** Part (a) guarantees that the specified output is possible. But the columns are dependent, so there are infinitely many inputs that produce it. In terms of a system, there are 7 unknowns and 5 pivots, leaving 2 free variables.

Now suppose B has shape 5×3 and rank 3.

d) (2 pts) Is every vector in $\mathbb{R}^5$ an output of B?

> **Solution:** **No.** $\dim(\text{colsp}(B)) = 3$, while $\dim(\mathbb{R}^5) = 5$. Thus $\text{colsp}(B)$ is a proper subspace of $\mathbb{R}^5$, and some vectors in $\mathbb{R}^5$ are not outputs of B.

e) (2 pts) Can two different inputs produce the same output under B?

> **Solution:** **No.** B has 3 columns and rank 3, so its columns are independent. Every output in $\text{colsp}(B)$ therefore comes from exactly one input. An output outside $\text{colsp}(B)$ has no input.
> The two matrices illustrate the two different comparisons: rank equal to the number of **rows** means every output is possible; rank equal to the number of **columns** means each possible output has a unique input.

Problem 5: Rows and columns (7 pts)

The first 3 columns of A below are linearly independent. Consider

$$A = \begin{bmatrix} 4 & 6 & -8 & 10 & -2 \\ 0 & 5 & 10 & -5 & 15 \\ 0 & 0 & 7 & 14 & -7 \end{bmatrix}, \qquad B = \begin{bmatrix} 4 & 0 & 0 \\ 6 & 5 & 0 \\ -8 & 10 & 7 \\ 10 & -5 & 14 \\ -2 & 15 & -7 \end{bmatrix}.$$

a) (4 pts) Find rank(A) and rank(B).

Solution: $\boxed{\text{rank}(A) = 3}$ because its first 3 columns are independent, and a subspace of $\mathbb{R}^3$ cannot have dimension greater than 3.
For B, call its columns $\vec{b}_1, \vec{b}_2, \vec{b}_3$. If $c_1\vec{b}_1 + c_2\vec{b}_2 + c_3\vec{b}_3 = \vec{0}$, the first 3 entries give

$$4c_1 = 0, \qquad 6c_1 + 5c_2 = 0, \qquad -8c_1 + 10c_2 + 7c_3 = 0.$$

The first equation forces $c_1 = 0$, the second then forces $c_2 = 0$, and the third forces $c_3 = 0$. Thus no column can be written as a combination of the other columns: the 3 columns are independent. Therefore $\boxed{\text{rank}(B) = 3}$.

b) (1 pt) How are the entries of A and B related?

Solution: The rows of A become the columns of B, in the same order. Equivalently, the entry in row i, column j of A becomes the entry in row j, column i of B.

c) (2 pts) Make a guess as to how the number of linearly independent rows of a matrix compares with the number of linearly independent columns.

Solution: The maximum number of linearly independent rows of a matrix appears to equal the maximum number of linearly independent columns. This is true in general. In this example, both numbers for A are 3: its rows are exactly the independent columns of B.

Problem 6: One subspace, several descriptions (21 pts)

Let

$$W = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathbb{R}^4 : x_3 = 4x_1 - 3x_2, \quad x_4 = 2x_1 + 5x_2 \right\}.$$

a) (5 pts) Write a general vector $\begin{bmatrix} a \\ b \\ 4a-3b \\ 2a+5b \end{bmatrix}$ in the subspace W as a linear combination of two linearly independent vectors. Use this to find a basis of W and determine $\dim(W)$.

Solution: Separate the contributions from a and b:

$$\begin{bmatrix} a \\ b \\ 4a-3b \\ 2a+5b \end{bmatrix} = a \underbrace{\begin{bmatrix} 1 \\ 0 \\ 4 \\ 2 \end{bmatrix}}_{\vec{u}} + b \underbrace{\begin{bmatrix} 0 \\ 1 \\ -3 \\ 5 \end{bmatrix}}_{\vec{v}}.$$

Every vector in W is such a combination, and every choice of a, b gives a vector satisfying the equations for W. Therefore $W = \text{span}(\vec{u}, \vec{v})$.

These vectors are not scalar multiples: their first 2 entries are $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Thus they are independent, and

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -3 \\ 5 \end{bmatrix} \right\} \text{ is a basis of } W, \qquad \dim(W) = 2.$$

For the remainder of this problem, let $\vec{u}, \vec{v}$ be your basis vectors from part (a). Furthermore, let

$$\vec{x} = 5\vec{u} + 9\vec{v}, \qquad \vec{y} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

b) (8 pts) Which of the following collections are bases of W? Explain why or why not for each collection.

(i) $\{\vec{u}+\vec{v},\ \vec{x}\}$ (ii) $\{\vec{x}-5\vec{u},\ \vec{v}\}$

(iii) $\{\vec{x},\ \vec{y}\}$ (iv) $\{4\vec{u}+7\vec{v},\ 7\vec{u}+12\vec{v}\}$.

Solution: We use the fact that $\vec{u}, \vec{v}$ form a basis of W. These arguments work for any basis chosen in part (a).

(i) **Yes.** Let $\vec{p} = \vec{u}+\vec{v}$ and $\vec{q} = \vec{x} = 5\vec{u}+9\vec{v}$. We can recover the original basis vectors:

$$\vec{v} = \frac{\vec{q}-5\vec{p}}{4}, \qquad \vec{u} = \frac{9\vec{p}-\vec{q}}{4}.$$

So the new pair spans W. It is independent: if $\vec{q} = c\vec{p}$, independence of $\vec{u}, \vec{v}$ would require both $c = 5$ and $c = 9$, which is impossible.

(ii) **No.** $\vec{x} - 5\vec{u} = 9\vec{v}$, so the pair is $\{9\vec{v}, \vec{v}\}$. It is dependent and spans only the line through $\vec{v}$, not all of W.

(iii) **No.** $\vec{y} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ is not in W: its fourth entry would have to be $2(0)+5(0) = 0$, but it is 1. A basis of W cannot contain a vector outside W.

(iv) **Yes.** Let $\vec{p} = 4\vec{u}+7\vec{v}$ and $\vec{q} = 7\vec{u}+12\vec{v}$. Then

$$\vec{u} = -12\vec{p}+7\vec{q}, \qquad \vec{v} = 7\vec{p}-4\vec{q}.$$

Thus the pair spans W. It is independent: $\vec{q} = c\vec{p}$ would require $c = 7/4$ and $c = 12/7$, which are different.

c) (8 pts) Find the rank of each matrix below. Explain your answers.

(i) $A = \begin{bmatrix} | & | \\ \vec{u} & \vec{v} \\ | & | \end{bmatrix}$, (ii) $B = \begin{bmatrix} | & | \\ \vec{u} & 2\vec{u} \\ | & | \end{bmatrix}$, (iii) $C = \begin{bmatrix} | & | & | \\ \vec{u} & \vec{v} & \vec{x} \\ | & | & | \end{bmatrix}$, (iv) $D = \begin{bmatrix} | & | & | & | \\ \vec{u} & \vec{v} & \vec{x} & \vec{y} \\ | & | & | & | \end{bmatrix}$.

Solution:

(i) $\boxed{\text{rank}(A) = 2}$. Its columns $\vec{u}, \vec{v}$ form a basis of W.

(ii) $\boxed{\text{rank}(B) = 1}$. $\vec{u}$ is nonzero, and $2\vec{u}$ adds no new direction. The column space is the line through $\vec{u}$.

(iii) $\boxed{\text{rank}(C) = 2}$. $\vec{x} = 5\vec{u}+9\vec{v}$ is redundant, so the column space is still W.

(iv) $\boxed{\text{rank}(D) = 3}$. $\vec{x}$ is redundant, but $\vec{y} \notin W$. Since $\vec{u}, \vec{v}$ span W, $\vec{y}$ cannot be made using them. Thus $\vec{u}, \vec{v}, \vec{y}$ form a basis of colsp(D).

Problem 7: Pivots, free variables, and zero outputs (27 pts)

Consider the factorization $A = LU$:

$$\underbrace{\begin{bmatrix} 4 & 8 & -4 & 0 \\ 16 & 32 & -11 & 15 \\ -12 & -24 & 37 & 75 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ -3 & 5 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 4 & 8 & -4 & 0 \\ 0 & 0 & 5 & 15 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{U}.$$

a) (2 pts) Write the three scalar equations in x_1, x_2, x_3, x_4 represented by $U\vec{x} = \vec{0}$.

Solution:

$$\boxed{\begin{aligned} 4x_1 + 8x_2 - 4x_3 &= 0, \\ 5x_3 + 15x_4 &= 0, \\ 0 &= 0. \end{aligned}}$$

The last equation imposes no restriction. The pivot variables are x_1, x_3; the free variables are x_2, x_4.

b) (4 pts) What is rank(A)? Explain in two ways:

(i) Using the pivots of U.

(ii) Using linear independence and a basis of colsp(A).

Solution: $\boxed{\text{rank}(A) = 2}$.

(i) U has 2 pivots, in columns 1 and 3. The number of pivots is the rank.

(ii) Call the columns of A $\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{a}_4$. They satisfy

$$\vec{a}_2 = 2\vec{a}_1, \qquad \vec{a}_4 = 3\vec{a}_1 + 3\vec{a}_3.$$

So $\vec{a}_1, \vec{a}_3$ span the column space. They are independent: their first entries would require $\vec{a}_3 = -\vec{a}_1$ if they were multiples, but their second entries are -11 and 16, and $-11 \neq -16$. Thus

$$\left\{ \begin{bmatrix} 4 \\ 16 \\ -12 \end{bmatrix}, \begin{bmatrix} -4 \\ -11 \\ 37 \end{bmatrix} \right\}$$

is a basis of colsp(A). This basis has 2 vectors, so $\dim(\text{colsp}(A)) = 2$.

c) (4 pts) Set $x_2 = s$ and $x_4 = t$. Use your equations from part (a) to fill in the blanks below, describing the set of inputs $\vec{x}$ such that $A\vec{x} = \vec{0}$:

$$\vec{x} = s \begin{bmatrix} ___ \\ 1 \\ ___ \\ 0 \end{bmatrix} + t \begin{bmatrix} ___ \\ 0 \\ ___ \\ 1 \end{bmatrix}, \qquad s, t \in \mathbb{R}.$$

Solution: Since L is invertible, $A\vec{x} = LU\vec{x} = \vec{0}$ holds exactly when $U\vec{x} = \vec{0}$. We can therefore use the equations from part (a).
Set $x_2 = s$ and $x_4 = t$. The second equation gives $5x_3 + 15t = 0$, so $x_3 = -3t$. The first then gives

$$4x_1 + 8s - 4(-3t) = 0 \quad \Longrightarrow \quad x_1 = -2s - 3t.$$

Thus the complete set of inputs sent to zero is

$$\boxed{\vec{x} = s \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 0 \\ -3 \\ 1 \end{bmatrix}, \qquad s, t \in \mathbb{R}.}$$

Every choice of s, t satisfies all 3 equations, and every solution has this form.

The set of inputs that A sends to zero is called its **null space**. It is itself a **subspace of** $\mathbb{R}^4$:

$$\text{nullsp}(A) = \{\vec{x} \in \mathbb{R}^4 : A\vec{x} = \vec{0}\}.$$

d) (2 pts) What is $\dim(\text{nullsp}(A))$? Explain using the 2 vectors in part (c).

Solution: Let

$$\vec{n}_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \vec{n}_2 = \begin{bmatrix} -3 \\ 0 \\ -3 \\ 1 \end{bmatrix}.$$

Part (c) shows that they span $\text{nullsp}(A)$. They are independent: they are not scalar multiples, as we can see from their second and fourth entries. They therefore form a basis, so $\boxed{\dim(\text{nullsp}(A)) = 2}$.
The null space consists of inputs in $\mathbb{R}^4$; the column space consists of outputs in $\mathbb{R}^3$. Both happen to have dimension 2 for this matrix, but they are different subspaces.

e) (5 pts) Show that nullsp(A) is a subspace: if $\vec{x}, \vec{y} \in$ nullsp(A) and $c \in \mathbb{R}$, show that $\vec{x} + \vec{y} \in$ nullsp(A) and $c\vec{x} \in$ nullsp(A).
Hint: You know $A\vec{x} = A\vec{y} = \vec{0}$. Use this to compute $A(\vec{x} + \vec{y})$ and $A(c\vec{x})$. Also check that $\vec{0} \in$ nullsp(A).

Solution: We check all 3 conditions for a subspace of $\mathbb{R}^4$.

1. **Contains zero.** $A\vec{0} = \vec{0}$, so $\vec{0} \in$ nullsp(A).
2. **Closed under addition.** If $\vec{x}, \vec{y} \in$ nullsp(A), then $A\vec{x} = \vec{0}$ and $A\vec{y} = \vec{0}$. By distributivity,
$$A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y} = \vec{0} + \vec{0} = \vec{0}.$$
Thus $\vec{x} + \vec{y} \in$ nullsp(A).
3. **Closed under scalar multiplication.** For any $c \in \mathbb{R}$ and $\vec{x} \in$ nullsp(A),
$$A(c\vec{x}) = c(A\vec{x}) = c\vec{0} = \vec{0}.$$
Thus $c\vec{x} \in$ nullsp(A).

All 3 conditions hold, so nullsp(A) is a subspace of $\mathbb{R}^4$.

f) (6 pts) Suppose, for this same A, that $A \begin{bmatrix} 9 \\ 1 \\ 2 \\ 6 \end{bmatrix} = \vec{y}$. Find three other inputs whose output is also $\vec{y}$. Then, in vector-parametric form, describe the complete set of vectors $\vec{v}$ such that $A\vec{v} = \vec{y}$.
Hint: If $A\vec{v} = \vec{y}$ and $\vec{n} \in$ nullsp(A)*, then what is $A(\vec{v} + \vec{n})$?*

Solution: Let $\vec{v}_0$ be the given input, so $A\vec{v}_0 = \vec{y}$. If $\vec{n} \in$ nullsp(A), then
$$A(\vec{v}_0 + \vec{n}) = A\vec{v}_0 + A\vec{n} = \vec{y} + \vec{0} = \vec{y}.$$
Using the null-space basis from part (d), we obtain 3 other inputs:
$$\vec{v}_0 + \vec{n}_1 = \boxed{\begin{bmatrix} 7 \\ 2 \\ 2 \\ 6 \end{bmatrix}}, \qquad \vec{v}_0 + \vec{n}_2 = \boxed{\begin{bmatrix} 6 \\ 1 \\ -1 \\ 7 \end{bmatrix}}, \qquad \vec{v}_0 + \vec{n}_1 + \vec{n}_2 = \boxed{\begin{bmatrix} 4 \\ 2 \\ -1 \\ 7 \end{bmatrix}}.$$
The complete set is
$$\boxed{\vec{v} = \begin{bmatrix} 9 \\ 1 \\ 2 \\ 6 \end{bmatrix} + s \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 0 \\ -3 \\ 1 \end{bmatrix}, \qquad s, t \in \mathbb{R}.}$$
Why does this give *every* input with output $\vec{y}$? If $A\vec{v} = \vec{y}$, then
$$A(\vec{v} - \vec{v}_0) = A\vec{v} - A\vec{v}_0 = \vec{y} - \vec{y} = \vec{0}.$$
Thus $\vec{v} - \vec{v}_0 \in$ nullsp(A), so it equals $s\vec{n}_1 + t\vec{n}_2$ for some s, t. No new system needs to be solved.

g) (4 pts) Complete the table below. Once the free variables are chosen, each pivot determines one remaining variable. Each free variable contributes one basis vector to the null space.

Matrix shape	Pivots (rank)	Free variables	Dimension of null space
3×4	2		
5×7	5		
$n \times d$	r		

Solution:

Matrix shape	Pivots (rank)	Free variables	Dimension of null space
3×4	2	2	2
5×7	5	2	2
$n \times d$	r	$d - r$	$d - r$

There are d input variables. Of these, r are pivot variables, so the remaining $d - r$ are free. In a vector-parametric description of the zero-output inputs, each free variable supplies a basis vector. Hence

$$\boxed{\text{rank}(A) + \dim(\text{nullsp}(A)) = d.}$$

The total is the number of *columns*, because that is the number of entries in each input.