Homework 8: Subspaces, Bases, and Rank

Math 124, Fall 2026 at the University of Michigan

due Friday, November 13th at 11:59PM

Write your solutions to the following problems either by writing them on a piece of paper or on a tablet and scanning your answers as a PDF. Note that you are not allowed to use LaTeX, Google Docs, or any other digital document creation software to type your answers. Homeworks are due to Pensive by 11:59PM on the due date. See the syllabus for details on the slip day policy.

Homework will be evaluated not only on the correctness of your answers, but on your ability to present your ideas clearly and logically. You should always explain and justify your conclusions, using sound reasoning. Your goal should be to convince the reader of your assertions. If a question does not require explanation, it will be explicitly stated.

Before proceeding, make sure you're familiar with the collaboration policy.

Total Points: $6 + 8 + 21 + 10 + 7 + 21 + 27 = 100$.

Note: Refer to **Chapter 5** while working on the homework!

Problem 1: Homework 7 Solutions Review (6 pts)

Review the solutions to Homework 7. Pick **two problem parts** (for example, Problem 3a and Problem 5b) from Homework 7 in which your solutions have the most room for improvement, i.e., where they have unsound reasoning, could be significantly more efficient or clearer, etc. **Include a screenshot of your solution to each problem part**, and in a few sentences, explain what was deficient and how it could be fixed.

Alternatively, if you think one of your solutions is significantly better than the posted one, copy it here and explain why you think it is better. If you didn't do Homework 7, choose two problem parts from it that look challenging to you, and in a few sentences, explain the key ideas behind their solutions in your own words.

Problem 2: Multiplying a column by a row (8 pts)

Let $A = \begin{bmatrix} 5 \\ 1 \\ 3 \\ -1 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 0 & 10 & -2 & 1 \end{bmatrix}$.

a) (4 pts) Write AB in matrix form. *Hint: The result is a* 4×5 *matrix.*

b) (2 pts) Fill in the blanks: Every column of AB is a scalar multiple of _____, while every row of AB is a scalar multiple of _____.

c) (2 pts) Find rank(AB).

Problem 3: Column spaces and rank (21 pts)

Consider the following matrices.

$$A = \begin{bmatrix} 4 & -2 \\ 10 & 0 \\ 1 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 3 & -4 \\ 4 & 3 \\ 0 & 0 \end{bmatrix}, \qquad C = \begin{bmatrix} 6 & 2 \\ 10 & 10 \\ 0 & 2 \end{bmatrix}.$$

$$D = \begin{bmatrix} 4 & -2 & 2 \\ 10 & 0 & 10 \\ 1 & 1 & 2 \end{bmatrix}, \quad E = \begin{bmatrix} 3 & -4 & 0 \\ 4 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}, \quad F = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 5 & 10 & -5 & 15 \\ 1 & 2 & -1 & 3 \end{bmatrix}, \quad G = \begin{bmatrix} 4 & -2 & 6 & -4 \\ 10 & 0 & 10 & 0 \\ 1 & 1 & 0 & 2 \end{bmatrix}.$$

a) (14 pts) For each matrix, determine whether its column space is exactly the plane $5x - 3y + 10z = 0$ in $\mathbb{R}^3$.

b) (7 pts) Find the rank of each matrix.

Problem 4: Reachable outputs and unique inputs (10 pts)

For parts (a)–(c), suppose A has shape 5×7 and rank 5. Determine whether each statement is **true or false**, and explain your answer.

a) (2 pts) $\text{colsp}(A) = \mathbb{R}^5$.

b) (2 pts) The columns of A are linearly independent.

c) (2 pts) There is exactly one input $\vec{x}$ such that $A\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$.

Now suppose B has shape 5×3 and rank 3.

d) (2 pts) Is every vector in $\mathbb{R}^5$ an output of B?

e) (2 pts) Can two different inputs produce the same output under B?

Problem 5: Rows and columns (7 pts)

The first 3 columns of A below are linearly independent. Consider

$$A = \begin{bmatrix} 4 & 6 & -8 & 10 & -2 \\ 0 & 5 & 10 & -5 & 15 \\ 0 & 0 & 7 & 14 & -7 \end{bmatrix}, \qquad B = \begin{bmatrix} 4 & 0 & 0 \\ 6 & 5 & 0 \\ -8 & 10 & 7 \\ 10 & -5 & 14 \\ -2 & 15 & -7 \end{bmatrix}.$$

a) (4 pts) Find $\text{rank}(A)$ and $\text{rank}(B)$.

b) (1 pt) How are the entries of A and B related?

c) (2 pts) Make a guess as to how the number of linearly independent rows of a matrix compares with the number of linearly independent columns.

Problem 6: One subspace, several descriptions (21 pts)

Let

$$W = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathbb{R}^4 : x_3 = 4x_1 - 3x_2, \quad x_4 = 2x_1 + 5x_2 \right\}.$$

a) (5 pts) Write a general vector $\begin{bmatrix} a \\ b \\ 4a-3b \\ 2a+5b \end{bmatrix}$ in the subspace W as a linear combination of two linearly independent vectors. Use this to find a basis of W and determine $\dim(W)$.

For the remainder of this problem, let $\vec{u}, \vec{v}$ be your basis vectors from part (a). Furthermore, let

$$\vec{x} = 5\vec{u} + 9\vec{v}, \qquad \vec{y} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

b) (8 pts) Which of the following collections are bases of W? Explain why or why not for each collection.

(i) $\{\vec{u} + \vec{v},\ \vec{x}\}$ (ii) $\{\vec{x} - 5\vec{u},\ \vec{v}\}$

(iii) $\{\vec{x},\ \vec{y}\}$ (iv) $\{4\vec{u} + 7\vec{v},\ 7\vec{u} + 12\vec{v}\}$.

c) (8 pts) Find the rank of each matrix below. Explain your answers.

(i) $A = \begin{bmatrix} | & | \\ \vec{u} & \vec{v} \\ | & | \end{bmatrix}$, (ii) $B = \begin{bmatrix} | & | \\ \vec{u} & 2\vec{u} \\ | & | \end{bmatrix}$, (iii) $C = \begin{bmatrix} | & | & | \\ \vec{u} & \vec{v} & \vec{x} \\ | & | & | \end{bmatrix}$, (iv) $D = \begin{bmatrix} | & | & | & | \\ \vec{u} & \vec{v} & \vec{x} & \vec{y} \\ | & | & | & | \end{bmatrix}$.

Problem 7: Pivots, free variables, and zero outputs (27 pts)

Consider the factorization $A = LU$:

$$\underbrace{\begin{bmatrix} 4 & 8 & -4 & 0 \\ 16 & 32 & -11 & 15 \\ -12 & -24 & 37 & 75 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ -3 & 5 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 4 & 8 & -4 & 0 \\ 0 & 0 & 5 & 15 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{U}.$$

a) (2 pts) Write the three scalar equations in x_1, x_2, x_3, x_4 represented by $U\vec{x} = \vec{0}$.

b) (4 pts) What is rank(A)? Explain in two ways:

(i) Using the pivots of U.

(ii) Using linear independence and a basis of colsp(A).

c) (4 pts) Set $x_2 = s$ and $x_4 = t$. Use your equations from part (a) to fill in the blanks below, describing the set of inputs $\vec{x}$ such that $A\vec{x} = \vec{0}$:

$$\vec{x} = s \begin{bmatrix} ___ \\ 1 \\ ___ \\ 0 \end{bmatrix} + t \begin{bmatrix} ___ \\ 0 \\ ___ \\ 1 \end{bmatrix}, \qquad s, t \in \mathbb{R}.$$

The set of inputs that A sends to zero is called its **null space**. It is itself a **subspace of** $\mathbb{R}^4$:

$$\text{nullsp}(A) = \{\vec{x} \in \mathbb{R}^4 : A\vec{x} = \vec{0}\}.$$

d) (2 pts) What is dim(nullsp(A))? Explain using the 2 vectors in part (c).

e) (5 pts) Show that nullsp(A) is a subspace: if $\vec{x}, \vec{y} \in$ nullsp(A) and $c \in \mathbb{R}$, show that $\vec{x} + \vec{y} \in$ nullsp(A) and $c\vec{x} \in$ nullsp(A).
Hint: You know $A\vec{x} = A\vec{y} = \vec{0}$. Use this to compute $A(\vec{x} + \vec{y})$ and $A(c\vec{x})$. Also check that $\vec{0} \in$ nullsp(A).

f) (6 pts) Suppose, for this same A, that $A \begin{bmatrix} 9 \\ 1 \\ 2 \\ 6 \end{bmatrix} = \vec{y}$. Find three other inputs whose output is also $\vec{y}$. Then, in vector-parametric form, describe the complete set of vectors $\vec{v}$ such that $A\vec{v} = \vec{y}$.
Hint: If $A\vec{v} = \vec{y}$ and $\vec{n} \in$ nullsp(A)*, then what is $A(\vec{v} + \vec{n})$?*

g) (4 pts) Complete the table below. Once the free variables are chosen, each pivot determines one remaining variable. Each free variable contributes one basis vector to the null space.

Matrix shape	Pivots (rank)	Free variables	Dimension of null space
3×4	2		
5×7	5		
$n \times d$	r		