Lab 8: Inverses Solutions

Math 124, Fall 2026 at the University of Michigan

due by the end of class on Wednesday, October 21, 2026

Name: _____

uniqname: _____

Each lab worksheet will contain several activities, most of which will involve writing math on paper, and some of which will involve running code in a Jupyter Notebook. To receive credit for today's lab, you must show your lab TA your work on this worksheet.

While you must get checked off by your lab TA **individually**, we encourage you to form groups with 1-2 other students to complete the activities together.

Review: Inverses. For a square matrix A, an **inverse** is a matrix A^{-1} satisfying

$$A^{-1}A = AA^{-1} = I.$$

Thus $A^{-1}(A\vec{x}) = \vec{x}$ for every input $\vec{x}$. A square matrix with an inverse is **invertible**; one without an inverse is **singular**.

Activity 1: A quick check

Let $D = \begin{bmatrix} -2 & 0 \\ 0 & 3 \end{bmatrix}$ and $E = \begin{bmatrix} -1/2 & 0 \\ 0 & 1/3 \end{bmatrix}$. Verify that $E = D^{-1}$ by computing both products in the definition above. Explain geometrically how E undoes D.

Solution: Both products are the identity:

$$DE = \begin{bmatrix} (-2)(-1/2) & 0 \\ 0 & 3(1/3) \end{bmatrix} = I, \qquad ED = \begin{bmatrix} (-1/2)(-2) & 0 \\ 0 & (1/3)3 \end{bmatrix} = I.$$

So $\boxed{E = D^{-1}}$. The matrix D flips the sign of the horizontal coordinate and stretches it by a factor of 2, and stretches the vertical coordinate by a factor of 3. The matrix E flips the horizontal sign back and scales the coordinates by $1/2$ and $1/3$, undoing both stretches.

Activity 2: Undo the motion

Find the inverse of each matrix below **without** using the formula given in Activity 3. Instead, use geometry and/or your understanding of the given transformation. Verify your answers by multiplying.

a) The matrix $Q = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ rotates counterclockwise by $90°$. Describe the inverse transformation and write Q^{-1}.

Solution: To undo a counterclockwise rotation by $90°$, rotate clockwise by $90°$. This sends $\begin{bmatrix} x \\ y \end{bmatrix}$ to $\begin{bmatrix} y \\ -x \end{bmatrix}$, so

$$\boxed{Q^{-1} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}.$$

Checking both orders,

$$QQ^{-1} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = I, \qquad Q^{-1}Q = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = I.$$

b) The shear $S = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ sends $\begin{bmatrix} x \\ y \end{bmatrix}$ to $\begin{bmatrix} x + 2y \\ y \end{bmatrix}$. Explain how to recover the original coordinates from the output, and use this to find S^{-1}.

Solution: The second coordinate doesn't change. To recover the first coordinate, subtract twice the second coordinate from the first. In other words, the inverse sends $\begin{bmatrix} u \\ v \end{bmatrix}$ to $\begin{bmatrix} u - 2v \\ v \end{bmatrix}$, so

$$\boxed{S^{-1} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}}.$$

Indeed,

$$SS^{-1} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -2+2 \\ 0 & 1 \end{bmatrix} = I,$$

$$S^{-1}S = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2-2 \\ 0 & 1 \end{bmatrix} = I.$$

Activity 3: A formula for 2×2 inverses

For a 2×2 matrix,

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \quad \text{given } ad - bc \neq 0.$$

The number $ad - bc$ is called the **determinant** of A. If $ad - bc = 0$, then A has no inverse.

a) Find the inverse of $A = \begin{bmatrix} 4 & -2 \\ 3 & 2 \end{bmatrix}$. Verify your result.

Solution: Here $ad - bc = 4(2) - (-2)(3) = 14$, so

$$\boxed{A^{-1} = \frac{1}{14} \begin{bmatrix} 2 & 2 \\ -3 & 4 \end{bmatrix}}.$$

Checking both products,

$$AA^{-1} = \frac{1}{14} \begin{bmatrix} 8+6 & 8-8 \\ 6-6 & 6+8 \end{bmatrix} = I, \qquad A^{-1}A = \frac{1}{14} \begin{bmatrix} 8+6 & -4+4 \\ -12+12 & 6+8 \end{bmatrix} = I.$$

b) For which values of t is $M = \begin{bmatrix} 4 & -2 \\ 3 & t \end{bmatrix}$ invertible? For the value of t that makes M singular, describe the span of its columns and explain why some output vectors cannot be reached.

Solution: The determinant is $4t - (-2)(3) = 4t + 6$. Thus, M is invertible exactly when

$$\boxed{t \neq -\frac{3}{2}}.$$

When $t = -3/2$, the second column is $-1/2$ times the first. Their span is therefore the line

$$\text{span}\left(\begin{bmatrix} 4 \\ 3 \end{bmatrix} \right).$$

Every output $M\vec{x}$ is a linear combination of these columns, so it lies on this line. A vector outside the line, such as $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, cannot be an output. There is no way to undo M on all of $\mathbb{R}^2$.

Activity 4: Recover the input

Consider the equation

$$A\vec{x} = \vec{b},$$

where

$$A = \begin{bmatrix} 4 & -2 \\ 3 & 2 \end{bmatrix}, \qquad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad \vec{b} = \begin{bmatrix} 6 \\ 8 \end{bmatrix}.$$

You found the inverse of A in Activity 3.

a) Write $A\vec{x} = \vec{b}$ as a system of two equations in the two unknowns x_1 and x_2. Then solve the system using A^{-1} and verify your solution.

Solution: Multiplying out $A\vec{x}$ gives the system

$$\begin{aligned} 4x_1 - 2x_2 &= 6, \\ 3x_1 + 2x_2 &= 8. \end{aligned}$$

Multiplying both sides of $A\vec{x} = \vec{b}$ on the left by A^{-1} gives $\vec{x} = A^{-1}\vec{b}$. Using our inverse from Activity 3,

$$\vec{x} = \frac{1}{14}\begin{bmatrix} 2 & 2 \\ -3 & 4 \end{bmatrix}\begin{bmatrix} 6 \\ 8 \end{bmatrix} = \frac{1}{14}\begin{bmatrix} 12+16 \\ -18+32 \end{bmatrix} = \boxed{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}.$$

Checking in the original equations, $4(2) - 2(1) = 6$ and $3(2) + 2(1) = 8$, as needed.

b) Suppose the first entry of $\vec{b}$ increases by 1 while the second stays fixed. How do x_1 and x_2 change? Explain how your answer is encoded in a column of A^{-1} and why the change does not depend on the original entries of $\vec{b}$.

Solution: Increasing the first entry of $\vec{b}$ by 1 adds $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ to $\vec{b}$. The new solution is

$$A^{-1}\left(\vec{b} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = A^{-1}\vec{b} + A^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \vec{x} + \frac{1}{14}\begin{bmatrix} 2 \\ -3 \end{bmatrix}.$$

So $\boxed{x_1 \text{ increases by } 1/7}$ and $\boxed{x_2 \text{ decreases by } 3/14}$. Multiplying by $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ picks out the first column of A^{-1}. The change depends only on this column and the change in $\vec{b}$, not on the original entries of $\vec{b}$.

Activity 5: Undo a sequence

An image editor first stretches horizontally by a factor of 2, then applies a horizontal shear:

$$D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \qquad S = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Find the combined matrix C and its inverse. Express C^{-1} as a product involving D^{-1} and S^{-1}, and explain the order in terms of undoing the two edits.

Solution: Stretching first and then shearing gives $S(D\vec{x})$, so

$$C = SD = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}}.$$

To undo the edits, first undo the shear, then undo the stretch. Since the matrix on the right acts first,

$$C^{-1} = D^{-1}S^{-1} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} 1/2 & -1/2 \\ 0 & 1 \end{bmatrix}}.$$

This order also works algebraically:

$$(D^{-1}S^{-1})(SD) = D^{-1}(S^{-1}S)D = I,$$

$$(SD)(D^{-1}S^{-1}) = S(DD^{-1})S^{-1} = I.$$

Activity 6: Solving matrix equations

a) Suppose A is a square matrix satisfying

$$A^4 - 3A^3 + A^2 + 3A + 5I = 0,$$

where I and 0 are the identity and zero matrices of the same shape as A. Find an expression for A^{-1} in terms of powers of A, and justify why it is an inverse.

Solution: Move the constant term to the other side and factor out A:

$$A^4 - 3A^3 + A^2 + 3A = -5I, \qquad A(A^3 - 3A^2 + A + 3I) = -5I.$$

Dividing by -5 gives

$$A\left[-\frac{1}{5}(A^3 - 3A^2 + A + 3I)\right] = I.$$

We can also factor A on the right, because multiplying powers of A in either order gives the same result. Thus the proposed inverse multiplies with A to give I in both orders, and

$$\boxed{A^{-1} = -\frac{1}{5}A^3 + \frac{3}{5}A^2 - \frac{1}{5}A - \frac{3}{5}I}.$$

b) Suppose $A = BC$, where

$$A = \begin{bmatrix} 5 & -3 \\ 10 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}.$$

Find C.

Solution: Multiply both sides of $A = BC$ on the left by B^{-1}:

$$B^{-1}A = B^{-1}BC = C.$$

Since $2(3) - 1(1) = 5$,

$$B^{-1} = \frac{1}{5}\begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}.$$

Therefore,

$$C = \frac{1}{5}\begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 5 & -3 \\ 10 & 1 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 5 & -10 \\ 15 & 5 \end{bmatrix} = \boxed{\begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}}.$$

As a check,

$$BC = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 5 & -3 \\ 10 & 1 \end{bmatrix} = A.$$