Lab 11: Gram-Schmidt and QR Factorization Solutions

Math 124, Fall 2026 at the University of Michigan

due by the end of class on Wednesday, November 11, 2026

Name: _____

uniqname: _____

Each lab worksheet will contain several activities, most of which will involve writing math on paper, and some of which will involve running code in a Jupyter Notebook. Lab activities are meant to last an hour, and the second hour of lab is dedicated to starting the homework assignment. To receive credit for a lab, you must show your lab TA your work on both the lab worksheet and homework assignment.

While you must get checked off by your lab TA **individually**, we encourage you to form groups with 1-2 other students to complete the activities together.

Activity 1: A 2×2 QR factorization

Consider the matrix

$$A = \begin{bmatrix} 5 & -7 \\ 12 & 17 \end{bmatrix}.$$

Call its columns $\vec{v}_1, \vec{v}_2$, in that order. Our goal is to find a QR factorization of A.

a) First, use the Gram-Schmidt process to find an orthonormal basis $\vec{u}_1, \vec{u}_2$ of $\text{colsp}(A)$.

Solution: The first column has length $\|\vec{v}_1\| = \sqrt{5^2 + 12^2} = 13$, so

$$\vec{u}_1 = \frac{1}{13}\begin{bmatrix} 5 \\ 12 \end{bmatrix}, \qquad \vec{v}_2 \cdot \vec{u}_1 = \frac{-35 + 204}{13} = 13.$$

Then

$$\vec{w}_2 = \vec{v}_2 - 13\vec{u}_1 = \begin{bmatrix} -7 \\ 17 \end{bmatrix} - \begin{bmatrix} 5 \\ 12 \end{bmatrix} = \begin{bmatrix} -12 \\ 5 \end{bmatrix}.$$

Its length is also 13, giving $\vec{u}_2 = \frac{1}{13}\begin{bmatrix} -12 \\ 5 \end{bmatrix}$.

The Gram-Schmidt process preserves the span: $\vec{u}_1, \vec{u}_2$ are combinations of the original columns, and the equations in part (b) recover those columns. Thus they form an orthonormal basis of $\text{colsp}(A)$.

b) Write the relationships between the original vectors ($\vec{v}_1$ and $\vec{v}_2$) and the orthonormal basis vectors ($\vec{u}_1$ and $\vec{u}_2$). Use these equations to find Q and R such that $A = QR$.

Solution: Undoing the normalization and projection steps gives the relationships $\vec{v}_1 = 13\vec{u}_1$ and $\vec{v}_2 = 13\vec{u}_1 + 13\vec{u}_2$. Thus

$$Q = \frac{1}{13}\begin{bmatrix} 5 & -12 \\ 12 & 5 \end{bmatrix}, \qquad R = \begin{bmatrix} 13 & 13 \\ 0 & 13 \end{bmatrix}.$$

Each column of R records the coefficients of one original vector in the orthonormal basis; the columns of Q are $\vec{u}_1$ and $\vec{u}_2$.

c) Verify that $QR = A$ and that the columns of Q are orthonormal.

Solution: Both columns have squared length $(25 + 144)/169 = 1$, and their dot product is $(-60 + 60)/169 = 0$. Also,

$$QR = \frac{1}{13}\begin{bmatrix} 65 & 65 - 156 \\ 156 & 156 + 65 \end{bmatrix} = \begin{bmatrix} 5 & -7 \\ 12 & 17 \end{bmatrix} = A.$$

Gram-Schmidt with three vectors

We'll now consider the case of three vectors. Suppose that A is a matrix whose columns $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly independent. We want to use the Gram-Schmidt method to construct an orthonormal basis $\vec{u}_1, \vec{u}_2, \vec{u}_3$ of their span and find the QR factorization of A. The first two steps are just what we've already done.

- **First vector: normalize.** Divide by its length:

$$\vec{u}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|}.$$

- **Second vector: subtract one projection, then normalize.** Let $\ell_1 = \text{span}(\vec{u}_1)$. Remove the part of $\vec{v}_2$ along this line:

$$\vec{w}_2 = \vec{v}_2 - \text{proj}_{\ell_1}(\vec{v}_2) = \vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1)\vec{u}_1, \qquad \vec{u}_2 = \frac{\vec{w}_2}{\|\vec{w}_2\|}.$$

- **Third vector: subtract two projections, then normalize.** Here's the new step. We want a vector perpendicular to both $\vec{u}_1$ and $\vec{u}_2$. Let $\ell_2 = \text{span}(\vec{u}_2)$. Subtract the parts along both lines:

$$\begin{aligned}\vec{w}_3 &= \vec{v}_3 - \text{proj}_{\ell_1}(\vec{v}_3) - \text{proj}_{\ell_2}(\vec{v}_3) \\ &= \vec{v}_3 - (\vec{v}_3 \cdot \vec{u}_1)\vec{u}_1 - (\vec{v}_3 \cdot \vec{u}_2)\vec{u}_2, \qquad \vec{u}_3 = \frac{\vec{w}_3}{\|\vec{w}_3\|}.\end{aligned}$$

Because $\vec{u}_1$ and $\vec{u}_2$ are perpendicular, removing the component along one doesn't change the component along the other. What's left is perpendicular to both. The remaining vectors $\vec{w}_2, \vec{w}_3$ cannot be zero, because the original columns are independent.

How does this give $A = QR$? First record how the old vectors relate to the new ones. Put back the components we subtracted:

$$\begin{aligned}\vec{v}_1 &= \boxed{\|\vec{v}_1\|}\vec{u}_1, \\ \vec{v}_2 &= \boxed{(\vec{v}_2 \cdot \vec{u}_1)}\vec{u}_1 + \boxed{\|\vec{w}_2\|}\vec{u}_2, \\ \vec{v}_3 &= \boxed{(\vec{v}_3 \cdot \vec{u}_1)}\vec{u}_1 + \boxed{(\vec{v}_3 \cdot \vec{u}_2)}\vec{u}_2 + \boxed{\|\vec{w}_3\|}\vec{u}_3.\end{aligned}$$

These equations can be written together as

$$\underbrace{\begin{bmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ | & | & | \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} | & | & | \\ \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \\ | & | & | \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} \|\vec{v}_1\| & \vec{v}_2 \cdot \vec{u}_1 & \vec{v}_3 \cdot \vec{u}_1 \\ 0 & \|\vec{w}_2\| & \vec{v}_3 \cdot \vec{u}_2 \\ 0 & 0 & \|\vec{w}_3\| \end{bmatrix}}_{R}.$$

So Q contains the orthonormal vectors, and each column of R contains the coefficients for one original vector. The diagonal entries are the lengths we divided by; the entries above them are the dot products we used for the projections. Keep these numbers as you work!

Activity 2: A 3×3 QR factorization

Now consider the matrix

$$A = \begin{bmatrix} 2 & 5 & -2 \\ -2 & -2 & 5 \\ 1 & 4 & 5 \end{bmatrix}.$$

Our goal is to find a QR factorization of A. To start, we'll apply the Gram-Schmidt process to A. Call A's columns $\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$, in that order.

a) Find $\vec{u}_1$, then subtract the component of $\vec{v}_2$ along $\vec{u}_1$ and normalize to find $\vec{u}_2$.

Solution: The first column has length $\sqrt{2^2 + (-2)^2 + 1^2} = 3$, so

$$\vec{u}_1 = \frac{1}{3}\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, \qquad \vec{v}_2 \cdot \vec{u}_1 = \frac{10 + 4 + 4}{3} = 6.$$

Then

$$\vec{w}_2 = \begin{bmatrix} 5 \\ -2 \\ 4 \end{bmatrix} - 2\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}.$$

Its length is 3, so $\vec{u}_2 = \frac{1}{3}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$.

b) Subtract the components of $\vec{v}_3$ along both $\vec{u}_1$ and $\vec{u}_2$, then normalize to find $\vec{u}_3$.

Solution: The projection coefficients are

$$\vec{v}_3 \cdot \vec{u}_1 = \frac{-4 - 10 + 5}{3} = -3, \qquad \vec{v}_3 \cdot \vec{u}_2 = \frac{-2 + 10 + 10}{3} = 6.$$

Subtracting both components gives

$$\vec{w}_3 = \vec{v}_3 - (-3)\vec{u}_1 - 6\vec{u}_2 = \begin{bmatrix} -2 \\ 5 \\ 5 \end{bmatrix} + \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} - 2\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}.$$

Its length is 3, so $\vec{u}_3 = \frac{1}{3}\begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}$.

c) Write the relationships between the original vectors ($\vec{v}_1$, $\vec{v}_2$, and $\vec{v}_3$) and the orthonormal basis vectors ($\vec{u}_1$, $\vec{u}_2$, and $\vec{u}_3$). Use these equations to find Q and R such that $A = QR$. Verify that $QR = A$ and that the columns of Q are orthonormal.

Solution: Putting back the components we subtracted and undoing normalization gives

$$\vec{v}_1 = 3\vec{u}_1, \qquad \vec{v}_2 = 6\vec{u}_1 + 3\vec{u}_2, \qquad \vec{v}_3 = -3\vec{u}_1 + 6\vec{u}_2 + 3\vec{u}_3.$$

Thus

$$Q = \frac{1}{3}\begin{bmatrix} 2 & 1 & -2 \\ -2 & 2 & -1 \\ 1 & 2 & 2 \end{bmatrix}, \qquad R = \begin{bmatrix} 3 & 6 & -3 \\ 0 & 3 & 6 \\ 0 & 0 & 3 \end{bmatrix}.$$

Each column of Q has squared length $9/9 = 1$. The distinct-column dot products are $(2 - 4 + 2)/9 = 0$, $(-4 + 2 + 2)/9 = 0$, and $(-2 - 2 + 4)/9 = 0$. Finally,

$$QR = \frac{1}{3}\begin{bmatrix} 6 & 15 & -6 \\ -6 & -6 & 15 \\ 3 & 12 & 15 \end{bmatrix} = \begin{bmatrix} 2 & 5 & -2 \\ -2 & -2 & 5 \\ 1 & 4 & 5 \end{bmatrix} = A.$$