Lecture 14 Worksheet

Math 124, Fall 2026 at the University of Michigan

Some lectures will have an accompanying worksheet. We will do our best to include these problems directly in the notes as activities or worked examples.

Activity 1: Distributivity: Keep the factors in order

Let

$$A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}, \qquad C = \begin{bmatrix} -1 & 3 \\ 0 & 2 \end{bmatrix}.$$

a) Compute $B + C$, then compute $A(B + C)$.

b) Compute AB and AC, then add your results. Compare $AB + AC$ with part (a). Write the distributive identity illustrated by these calculations.

c) Without substituting the numerical matrices, expand

$$(A + B)^2 = (A + B)(A + B).$$

Use distributivity and keep every product in its original order. Is the result always $A^2 + 2AB + B^2$? Explain.

Activity 2: What can the same output tell us?

In this activity, A and B are 2×2 matrices, and all vectors belong to $\mathbb{R}^2$. For each claim, explain why it must be true or give a counterexample. A counterexample must include specific matrices or vectors and the relevant products.

a) Suppose $A\vec{x} = B\vec{x}$ for one nonzero vector $\vec{x}$. Must $A = B$?

b) Now suppose $A\vec{x} = B\vec{x}$ for **every** vector $\vec{x} \in \mathbb{R}^2$. Must $A = B$? Explain what this tells us about the columns of A and B.

c) Suppose $A\vec{x} = A\vec{y}$. Must $\vec{x} = \vec{y}$? What could a transformation do that would cause different inputs to have the same output?

Activity 3: Practice with inverses

For a 2×2 matrix, use

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}, \qquad ad - bc \neq 0.$$

If $ad - bc = 0$, the matrix has no inverse.

a) Find A^{-1} for $A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$. Check that $AA^{-1} = I_2$ and $A^{-1}A = I_2$.

b) Find B^{-1} for $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Check that $BB^{-1} = I_2$.

c) Does $C = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ have an inverse? Explain.

d) Let $\vec{v} = \begin{bmatrix} 5 \\ 11 \end{bmatrix}$. Compute $B^{-1}\vec{v}$, call your answer $\vec{x}$, and check that $B\vec{x} = \vec{v}$.

e) Suppose $B\vec{y} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Find $\vec{y}$ without doing another matrix multiplication. Which column of B^{-1} gives the answer? Explain.

Activity 4: Can we undo it?

Consider

$$H = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \qquad R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \qquad P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

a) Describe each transformation. Find H^{-1} and R^{-1} by thinking about how to undo their actions.

b) Find two different vectors that P sends to the same output. Explain why P cannot have an inverse.

c) We stretch using H, then rotate using R. Write the matrix for this combined transformation. Describe how to undo it, and compute its inverse.

d) Return to Activity 2(c). If A is invertible, prove that $A\vec{x} = A\vec{y}$ implies $\vec{x} = \vec{y}$.

Activity 5: Practice with transposes

Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \\ -1 & 1 \end{bmatrix}$ and $\vec{x} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$.

a) Find A^T and $\vec{x}^T$. Then compute $A^T\vec{x}$ and $\vec{x}^T A$. State the shape of each answer. What do you notice about the last two answers?

b) Which of $A\vec{x}$, $\vec{x}^T A^T$, and $\vec{x}\vec{x}^T$ are defined? Give the shape of each product that is defined. You do not need to compute the products.

c) Let $B = \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix}$. Compute AB, then transpose your answer.

d) Compute $B^T A^T$ and compare it with part (c). Is $A^T B^T$ defined? Complete the identity:

$$(AB)^T = _____.$$

e) Return to the matrices H and R from Activity 4. Is a matrix's transpose always its inverse? Use these two matrices to explain.