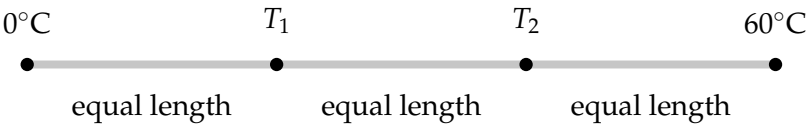
Lecture 15 Worksheet

Math 124, Fall 2026 at the University of Michigan

Some lectures will have an accompanying worksheet. We will do our best to include these problems directly in the notes as activities or worked examples.

Activity 1: Temperatures in a metal bar

An engineer wants to find the temperatures inside a metal bar. The left end is kept at $0°$C and the right end at $60°$C. The bar has three equal segments, with unknown temperatures T_1 and T_2 at the two interior points. No heat escapes through the sides.



Heat flows from right to left. Assume each segment has thermal conductance 1 watt per degree Celsius: the heat transferred per second, in watts, equals the temperature at its right end minus the temperature at its left end.

The temperatures have stopped changing, so at each interior point, heat arriving from the right equals heat leaving to the left.

a) At T_1, write expressions for the heat arriving and the heat leaving, and set them equal. Do the same at T_2. Simplify both equations.

b) Write your system as $A\vec{T} = \vec{b}$, where $\vec{T} = \begin{bmatrix} T_1 \\ T_2 \end{bmatrix}$. You will solve it in Activity 3.

Activity 2: Triangular systems

A square matrix is **upper triangular** if all entries below its main diagonal are zero, and **lower triangular** if all entries above its main diagonal are zero.

For each system, write a matrix equation and identify whether its coefficient matrix is upper or lower triangular. Then solve the system, explaining which variable you find first and why. Write your answer as a column vector.

a) $2x + y = 7,$
$y = 3.$

b) $2x = 2,$
$2x + 3y = 8,$
$x - y + 2z = 5.$

Starting at the bottom of an upper triangular system is called **back substitution**. Starting at the top of a lower triangular system is called **forward substitution**. When the diagonal entries are nonzero, each step determines one variable.

Activity 3: Finding and using an LU factorization

An **LU factorization** writes $A = LU$, where L is lower triangular and U is upper triangular. Throughout this worksheet, use **ones on the diagonal of** L.

a) Multiply the matrices below and match entries to find ℓ, p, q, r. Do not reorder the rows.

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ \ell & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} p & q \\ 0 & r \end{bmatrix}}_{U}.$$

Hint: Find the first row of U, then ℓ, then r.

b) Use your factors to solve the heat-flow system from Activity 1. Set $\vec{y} = U\vec{T}$, so that $LU\vec{T} = \vec{b}$ becomes two triangular systems. First solve $L\vec{y} = \vec{b}$, then solve $U\vec{T} = \vec{y}$.

c) Check your temperatures in the original equations. Do they make physical sense? Explain.

Activity 4: A three by three factorization

Find an LU factorization of

$$A = \begin{bmatrix} 3 & 6 & -3 \\ -4 & -5 & 10 \\ 5 & 8 & -4 \end{bmatrix}$$

without reordering its rows. Use the shapes

$$L = \begin{bmatrix} 1 & 0 & 0 \\ \ell_{21} & 1 & 0 \\ \ell_{31} & \ell_{32} & 1 \end{bmatrix}, \qquad U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}.$$

a) Match entries of LU to entries of A. Find the first row of U, the first column of L, the second row of U, the second column of L, and finally u_{33}. Show your calculations and write your completed factors.

b) Multiply your completed matrices L and U to check that $LU = A$.

Activity 5: A different coefficient matrix

Consider the system

$$\begin{bmatrix} 0 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \end{bmatrix}.$$

a) Try to find an LU factorization of the coefficient matrix using $L = \begin{bmatrix} 1 & 0 \\ \ell & 1 \end{bmatrix}$ and $U = \begin{bmatrix} p & q \\ 0 & r \end{bmatrix}$. Which entry of LU makes this impossible?

b) Reorder the equations so that the new coefficient matrix has an LU factorization. Write the reordered matrix equation, including its right-hand side, and find L and U.

c) Use these factors to solve the reordered system. Check your answer in the original system. Does the failure to find factors in part (a) mean the original system has no solution?

Activity 6: A general two by two matrix

Let A be invertible, and write

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad L = \begin{bmatrix} 1 & 0 \\ \ell & 1 \end{bmatrix}, \qquad U = \begin{bmatrix} p & q \\ 0 & r \end{bmatrix}.$$

a) Match entries in $A = LU$: first find the first row of U, then the first column of L, then the second row of U. At which step might you get stuck, and why?

b) For which invertible matrices does this procedure work without reordering the rows? Give formulas for L and U. Explain why both diagonal entries of U are nonzero and why the factors are uniquely determined.

c) If the procedure gets stuck, explain why swapping the rows allows you to continue. Use the invertibility of A to justify your answer, and find the factors for the reordered matrix.