Lecture 18 Worksheet

Math 124, Fall 2026 at the University of Michigan

Some lectures will have an accompanying worksheet. We will do our best to include these problems directly in the notes as activities or worked examples.

Activity 1: The column space

Consider the matrix

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Today's big question: What is the set of possible outputs of $A\vec{v}$?

a) Suppose we multiply A by $\vec{v} = \begin{bmatrix} a \\ b \end{bmatrix}$. What are the components of the output $A\vec{v}$?

b) Now let $\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be a proposed output. What relationship must x, y, and z satisfy for there to be an input $\vec{v}$ with $A\vec{v} = \vec{x}$?

c) Fill in the blanks: The column space of A, denoted colsp(A), is a _____ in $\mathbb{R}$—.

d) Which of the following vectors are in colsp(A)?

$$\begin{bmatrix} 3 \\ 5 \\ 2 \end{bmatrix}, \qquad \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}, \qquad \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}.$$

Activity 2: Adding a redundant column

Now consider the matrix B:

$$B = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}.$$

a) It turns out that B's third column is a linear combination of its first and second columns. Fill in the blanks:

$$\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = _____ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + _____ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

b) Suppose $A \begin{bmatrix} 4 \\ 5 \end{bmatrix} = \vec{y}$. Find a vector $\vec{v} \in \mathbb{R}^3$ such that

$$B\vec{v} = \vec{y}.$$

c) Fill in the blank: colsp(B) is _____.

d) Find two different inputs for B that produce $\begin{bmatrix} 2 \\ 5 \\ 3 \end{bmatrix}$: one with third entry zero and one with third entry one. Does knowing this output tell us which input was used?

e) Can the new column help us produce $\begin{bmatrix} 2 \\ 6 \\ 3 \end{bmatrix}$? Why or why not?

Activity 3: Adding a non-redundant column

Consider C:

$$C = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 4 \\ 0 & 1 & 3 \end{bmatrix}.$$

a) Notice that C's first two columns are the same as A's first two columns. C's third column is new. Is C's third column in colsp(A)? In other words, can it be written as a linear combination of A's columns? Why or why not?

b) Suppose $A \begin{bmatrix} 4 \\ 5 \end{bmatrix} = \vec{y}$. Find a vector $\vec{v} \in \mathbb{R}^3$ such that

$$C\vec{v} = \vec{y}.$$

c) Find a vector $\vec{x}$ such that

$$C\vec{x} = \begin{bmatrix} 2 \\ 6 \\ 3 \end{bmatrix}.$$

d) Find an LU factorization of C. What does this tell you about the number of solutions to $C\vec{x} = \vec{y}$, for any vector $\vec{y} \in \mathbb{R}^3$?

e) Fill in the blank: colsp(C) is _____.
Draw a Venn diagram relating colsp(A), colsp(B), and colsp(C).

f) Which of the following matrices are invertible? Explain your answers in the context of the number of inputs that produce the output $\begin{bmatrix} 2 \\ 5 \\ 3 \end{bmatrix}$.

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 4 \\ 0 & 1 & 3 \end{bmatrix}.$$