Lecture 19 Worksheet

Math 124, Fall 2026 at the University of Michigan

Some lectures will have an accompanying worksheet. We will do our best to include these problems directly in the notes as activities or worked examples.

Definition: Linearly independent. The collection of vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_d$ is **linearly independent** when none of the vectors in the collection can be written as a linear combination of the other vectors in the collection.

Fact: A square matrix is invertible if and only if its columns are linearly independent.

Activity 1: Recap

For each matrix, check the properties that apply and select the number of inputs that produce $\begin{bmatrix} 2 \\ 5 \\ 3 \end{bmatrix}$.

(i) $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$

☐ Linearly independent columns ☐ Invertible

Inputs for $\begin{bmatrix} 2 \\ 5 \\ 3 \end{bmatrix}$: ◯ 0 ◯ 1 ◯ Infinitely many

(ii) $B = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$

☐ Linearly independent columns ☐ Invertible

Inputs for $\begin{bmatrix} 2 \\ 5 \\ 3 \end{bmatrix}$: ◯ 0 ◯ 1 ◯ Infinitely many

(iii) $C = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 4 \\ 0 & 1 & 3 \end{bmatrix}$

☐ Linearly independent columns ☐ Invertible

Inputs for $\begin{bmatrix} 2 \\ 5 \\ 3 \end{bmatrix}$: ◯ 0 ◯ 1 ◯ Infinitely many

As we saw last class, colsp(A) and colsp(B) are both equal to the plane $x - y + z = 0$, while colsp$(C) = \mathbb{R}^3$.

Activity 2: Inverses

For each matrix below, determine whether or not it is invertible. Explain using its columns or the number of inputs that produce each possible output.

a)

$$H = \begin{bmatrix} 0 & 2 & 0 & 0 \\ -3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 5 & 0 \end{bmatrix}.$$

b)

$$J = \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & -3 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}.$$

Activity 3: Subspaces

A **subspace** generalizes a line or plane through the origin: it is a set of vectors that contains $\vec{0}$ and is "flat" in some sense. Formally, a subset S of $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$ if:

1. $\vec{0} \in S$.
2. If $\vec{u}, \vec{v} \in S$, then $\vec{u} + \vec{v} \in S$. We say S is **closed** under addition.
3. If $\vec{u} \in S$ and $c \in \mathbb{R}$, then $c\vec{u} \in S$. We say S is **closed** under scalar multiplication.

a) Why is the set $\left\{ \begin{bmatrix} x \\ y \end{bmatrix} : y = x^2,\ x \in \mathbb{R} \right\}$ not a subspace of $\mathbb{R}^2$?

b) What are all possible types of subspaces in $\mathbb{R}^2$?

c) What are all possible types of subspaces in $\mathbb{R}^3$?

Activity 4: Bases and dimension

Consider the 4×3 matrix

$$A = \begin{bmatrix} | & | & | \\ \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 5 \\ 0 & 1 & -3 \\ -1 & 3 & -10 \end{bmatrix}.$$

Let $\vec{a}_1$, $\vec{a}_2$, and $\vec{a}_3$ refer to the columns of A.

Fact: Every column space is a subspace.

a) Fill in the blank: colsp(A) is a subspace of $\mathbb{R}^{\square}$.

b) Express $\vec{a}_3$ as a linear combination of $\vec{a}_1$ and $\vec{a}_2$. Does removing $\vec{a}_3$ change the column space?

$$A = \begin{bmatrix} | & | & | \\ \vec{a}_1 & \vec{a}_2 & \vec{a}_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 5 \\ 0 & 1 & -3 \\ -1 & 3 & -10 \end{bmatrix}.$$

A **basis** of a subspace is a collection of linearly independent vectors that spans the subspace.

c) Which of the following collections are bases of colsp(A)? For each collection that is not a basis, explain what goes wrong.

$$\{\vec{a}_1\}, \qquad \{\vec{a}_1, \vec{a}_2\}, \qquad \{\vec{a}_1, \vec{a}_2, \vec{a}_3\}, \qquad \{\vec{a}_1, \vec{a}_1 + \vec{a}_2\}.$$

The **dimension** of a subspace is the number of vectors in a basis. Every basis of the same subspace contains the same number of vectors.

d) What is dim(colsp(A))? Describe colsp(A) geometrically.

Activity 5: Rank and solvability

The **rank** of a matrix is the dimension of its column space:

$$\boxed{\text{rank}(A) = \dim(\text{colsp}(A)).}$$

Think of the rank as the maximum number of linearly independent columns in the matrix.

a) Find the rank of each matrix below.

(i) $\begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 5 \\ 0 & 1 & -3 \\ -1 & 3 & -10 \end{bmatrix}$ rank = $\boxed{}$

(ii) $\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$ rank = $\boxed{}$

(iii) $\begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix}$ rank = $\boxed{}$

(iv) $\begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 4 \\ 0 & 1 & 3 \end{bmatrix}$ rank = $\boxed{}$

(v) $\begin{bmatrix} 1 & 0 & 2 & 0 \\ 1 & 1 & 4 & 0 \\ 0 & 1 & 3 & 1 \end{bmatrix}$ rank = $\boxed{}$

(vi) $\begin{bmatrix} 0 & 2 & 0 & 0 \\ -3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 5 & 0 \end{bmatrix}$ rank = $\boxed{}$

(vii) $\begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & -3 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}$ rank = $\boxed{}$

(viii) $\begin{bmatrix} 2 & -4 \\ -1 & 2 \\ 0 & 0 \\ 3 & -6 \\ 1 & -2 \end{bmatrix}$ rank = $\boxed{}$

b) For a 7×10 matrix A, what is its smallest possible rank? Largest possible rank?

Smallest: $\boxed{}$ Largest: $\boxed{}$

c) Fill in the blank: An $n \times n$ matrix is invertible if and only if $\text{rank}(A) =$ _____.

Rank and solvability. For an $n \times d$ matrix A:

- $A\vec{x} = \vec{b}$ has a solution exactly when $\vec{b} \in \text{colsp}(A)$.
- Every $\vec{b} \in \mathbb{R}^n$ is an output exactly when $\text{rank}(A) = n$.