Lecture 20 Worksheet

Math 124, Fall 2026 at the University of Michigan

Some lectures will have an accompanying worksheet. We will do our best to include these problems directly in the notes as activities or worked examples.

Activity 1: An orthonormal basis for a plane

The vectors

$$\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \qquad \vec{v}_2 = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$$

form a basis of the plane

$$V = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 : x + y - z = 0 \right\}.$$

a) Apply Gram–Schmidt to find an orthonormal basis $\vec{u}_1, \vec{u}_2$ of V. Check their lengths and dot product.

b) Use dot products to find coefficients c_1, c_2 such that

$$\vec{b} = \begin{bmatrix} 8 \\ -3 \\ 5 \end{bmatrix} = c_1\vec{u}_1 + c_2\vec{u}_2.$$

Check your answer by substituting the vectors back in.

Activity 2: Recording the result as a QR factorization

Use the original basis and the orthonormal basis from Activity 1. Let

$$A = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & -2 \\ 3 & 1 \end{bmatrix}.$$

a) Write $\vec{v}_1$ and $\vec{v}_2$ as combinations of $\vec{u}_1$ and $\vec{u}_2$. You can recover these relations from your Gram–Schmidt calculations.

b) Use these relations to write $A = QR$, where the columns of Q are $\vec{u}_1, \vec{u}_2$ and R is upper triangular.

c) Check that $QR = A$.

Activity 3: A 2×2 QR factorization

Find a QR factorization of

$$A = \begin{bmatrix} 5 & -7 \\ 12 & 17 \end{bmatrix}.$$

Call its columns $\vec{v}_1, \vec{v}_2$, in that order.

a) Use Gram–Schmidt to find an orthonormal basis $\vec{u}_1, \vec{u}_2$ of the column space. Remember to divide by the length after subtracting the projection.

b) Write the relations between the original vectors and the orthonormal vectors. Use these relations to find Q and R.

c) Check that the columns of Q are orthonormal and that $QR = A$.

Activity 4: A 3×3 QR factorization

Find a QR factorization of

$$A = \begin{bmatrix} 2 & 5 & -2 \\ -2 & -2 & 5 \\ 1 & 4 & 5 \end{bmatrix}.$$

Call its columns $\vec{v}_1, \vec{v}_2, \vec{v}_3$, in that order. Use Gram–Schmidt and keep your answers exact.

a) Find $\vec{u}_1$, then subtract the component of $\vec{v}_2$ along $\vec{u}_1$ and normalize to find $\vec{u}_2$.

b) Subtract the components of $\vec{v}_3$ along both $\vec{u}_1$ and $\vec{u}_2$, then normalize to find $\vec{u}_3$.

c) Write the relations between the original vectors and the orthonormal vectors, and assemble Q and R. Check that the columns of Q are orthonormal and that $QR = A$.