Lecture 22 Worksheet

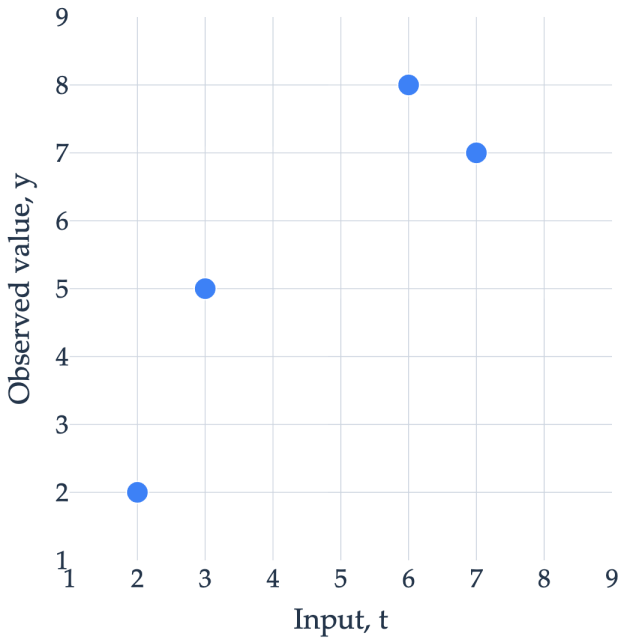
Math 124, Fall 2026 at the University of Michigan

> Some lectures will have an accompanying worksheet. We will do our best to include these problems directly in the notes as activities or worked examples.

Activity 1: Fitting four observations

We want to fit a line $y = c + mt$ to these four data points:

t	y
3	5
2	2
7	7
6	8



Unfortunately, no line passes through all four points. That's where least squares comes in: we want to find the line that fits the data *as best as possible!*

Write one equation for each data point, then write the four equations as $A\vec{x} = \vec{b}$, with $\vec{x} = \begin{bmatrix} c \\ m \end{bmatrix}$.

Activity 2: Finding a projection

Let A be an $n \times d$ matrix and let $\vec{b} \in \mathbb{R}^n$. We want to choose $\vec{x}^* \in \mathbb{R}^d$ so that $\vec{p} = A\vec{x}^*$ is the closest vector in colsp(A) to $\vec{b}$. Equivalently, we want to minimize

$$\|\vec{b} - A\vec{x}\|.$$

The closest vector $\vec{p}$ is the **projection of $\vec{b}$ onto** colsp(A).

a) Let $\vec{r} = \vec{b} - A\vec{x}$ be the error vector. What geometric property should the error vector $\vec{r}$ satisfy?

b) Write the columns of A as $\vec{a}_1, \ldots, \vec{a}_d$. Express the condition in (a) using dot products with these columns. Why is it enough to check the columns? Combine these conditions into one matrix equation involving A^T and $\vec{r}$.

c) Substitute $\vec{r} = \vec{b} - A\vec{x}$ into your matrix equation and rearrange it. Write the resulting equation with $\vec{x} = \vec{x}^*$. This is called the **normal equation**.

Activity 2: Finding a projection (continued)

d) Suppose A^TA is invertible. Then, what is the unique solution $\vec{x}^*$ to the normal equation?

Activity 3: A projection onto a subspace

Let

$$V = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}, \qquad \vec{b} = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}.$$

Find the projection of $\vec{b}$ onto V. Then, verify that the error vector is orthogonal to V.

Activity 4: Finding the best-fitting line

Return to the data in Activity 1. Use the normal equations to find the best-fitting line $y = c + mt$, then draw your line on the data below.

